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Research Article

Lightlike Hypersurfaces of a Semi-Riemannian Product Manifold and Quarter-Symmetric Nonmetric Connections

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We study lightlike hypersurfaces of a semi-Riemannian product manifold. We introduce a class of lightlike hypersurfaces called screen semi-invariant lightlike hypersurfaces and radical anti-invariant lightlike hypersurfaces. We consider lightlike hypersurfaces with respect to a quarter-symmetric nonmetric connection which is determined by the product structure. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifolds and the quarter-symmetric nonmetric connection, and we obtain some results.

1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of important topics of differential geometry. The geometry of lightlike submanifolds of a semi-Riemannian manifold, was presented in [1] (see also [2, 3]) by Duggal and Bejancu. In [4], Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [5], Kılıç and Şahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold and gave some examples and results for lightlike submanifolds. The lightlike hypersurfaces have been studied by many authors in various spaces (for example [6, 7]).

In [8], Hayden introduced a metric connection with nonzero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semisymmetric (symmetric) and nonmetric connection have been studied by many authors [9–14]. In [15], Yaşar et al. have studied lightlike hypersurfaces in semi-Riemannian manifolds with semisymmetric nonmetric connection. The idea of quarter-symmetric linear connections in a differential manifold was introduced by Golab [11]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\overline{T}$ is of the form:

$$\overline{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \tag{1.1}$$

for any vector fields X, Y on a manifold, where u is a 1-form and φ is a tensor of type (1,1).

In this paper, we study lightlike hypersurfaces of a semi-Riemannian product manifold. As a first step, in Section 3, we introduce screen semi-invariant lightlike hypersurfaces and radical anti-invariant lightlike hypersurfaces of a semi-Riemannian product manifold. We give some examples and study their geometric properties. In Section 4, we consider lightlike hypersurfaces of a semi-Riemannian product manifold with quarter-symmetric nonmetric connection determined by the product structure. We compute the Riemannian curvature tensor with respect to the quarter-symmetric nonmetric connection and give some results.

2. Lightlike Hypersurfaces

Let $(\overline{M}, \overline{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold with index $(\overline{g}) = q \geq 1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. If the induced metric g on M is degenerate, then M is called a lightlike (null or degenerate) hypersurface [1] (see also [2, 3]). Then there exists a null vector field $\xi \neq 0$ on M such that

$$g(\xi, X) = 0, \quad \forall X \in \Gamma(TM). \tag{2.1}$$

The radical or the null space of T_xM, at each point $x \in M$, is a subspace Rad T_xM defined by

$$\text{Rad } T_xM = \left\{ \xi \in T_xM \mid g_x(\xi, X) = 0, \forall X \in \Gamma(TM) \right\}, \tag{2.2}$$

whose dimension is called the nullity degree of g. We recall that the nullity degree of g for a lightlike hypersurface of $\overline{M}$ is 1. Since g is degenerate and any null vector being perpendicular to itself, $T_xM^{\perp}$ is also null and

$$\text{Rad } T_xM = T_xM \cap T_xM^{\perp}. \tag{2.3}$$

Since $\dim T_xM^{\perp} = 1$ and $\dim \text{Rad } T_xM = 1$, we have $\text{Rad}\, T_xM = T_xM^{\perp}$. We call Rad TM a radical distribution and it is spanned by the null vector field ξ. The complementary vector bundle $S(TM)$ of Rad TM in TM is called the screen bundle of M. We note that any screen bundle is nondegenerate. This means that

$$TM = \text{Rad } TM \perp S(TM). \tag{2.4}$$

Here $\perp$ denotes the orthogonal-direct sum. The complementary vector bundle $S(TM)^{\perp}$ of $S(TM)$ in $T\overline{M}$ is called screen transversal bundle and it has rank 2. Since Rad TM is a lightlike subbundle of $S(TM)^{\perp}$ there exists a unique local section N of $S(TM)^{\perp}$ such that

$$\overline{g}(N,N)=0, \qquad \overline{g}(\xi,N)=1. \tag{2.5}$$

Note that N is transversal to M and $\{\xi,N\}$ is a local frame field of $S(TM)^{\perp}$ and there exists a line subbundle $\operatorname{ltr}(TM)$ of $T\overline{M}$, and it is called the lightlike transversal bundle, locally spanned by N. Hence we have the following decomposition:

$$T\overline{M}=TM\oplus \operatorname{ltr}(TM)=S(TM)\perp \operatorname{Rad} TM\oplus \operatorname{ltr}(TM), \tag{2.6}$$

where $\oplus$ is the direct sum but not orthogonal [1, 3]. From the above decomposition of a semi-Riemannian manifold $\overline{M}$ along a lightlike hypersurface M, we can consider the following local quasiorthonormal field of frames of $\overline{M}$ along M:

$$\{X_1,\ldots,X_m,\xi,N\}, \tag{2.7}$$

where $\{X_1,\ldots,X_m\}$ is an orthonormal basis of $\Gamma(S(TM))$. According to the splitting (2.6), we have the following Gauss and Weingarten formulas, respectively:

$$\begin{aligned}\overline{\nabla}_XY&=\nabla_XY+h(X,Y),\\ \overline{\nabla}_XN&=-A_NX+\nabla^t_XN,\end{aligned} \tag{2.8}$$

for any $X,Y\in\Gamma(TM)$, where $\nabla_XY, A_NX\in\Gamma(TM)$ and $h(X,Y),\nabla^t_XN\in\Gamma(\operatorname{ltr}(TM))$. If we set $B(X,Y)=\overline{g}(h(X,Y),\xi)$ and $\tau(X)=\overline{g}(\nabla^t_XN,\xi)$, then (2.8) become

$$\overline{\nabla}_XY=\nabla_XY+B(X,Y)N, \tag{2.9}$$

$$\overline{\nabla}_XN=-A_NX+\tau(X)N. \tag{2.10}$$

B and A are called the second fundamental form and the shape operator of the lightlike hypersurface M, respectively [1]. Let P be the projection of $S(TM)$ on M. Then, for any $X\in\Gamma(TM)$, we can write

$$X=PX+\eta(X)\xi, \tag{2.11}$$

where η is a 1-form given by

$$\eta(X)=\overline{g}(X,N). \tag{2.12}$$

From (2.9), we get

$$(\nabla_Xg)(Y,Z)=B(X,Y)\eta(Z)+B(X,Z)\eta(Y), \quad \forall X,Y,Z\in\Gamma(TM), \tag{2.13}$$

and the induced connection ∇ is a nonmetric connection on M. From (2.4), we have

$$\begin{aligned}\nabla_X W &= \nabla^*_X W + h^*(X,W)\\ &= \nabla^*_X W + C(X,W)\xi, \quad X\in\Gamma(TM), W\in\Gamma(S(TM)),\\ \nabla_X \xi &= -A^*_\xi X - \tau(X)\xi,\end{aligned} \tag{2.14}$$

where $\nabla^*_X W$ and $A^*_\xi X$ belong to $\Gamma(S(TM))$. C, A^*_ξ and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$, respectively. Also, we have the following identities:

$$\begin{gathered} g\left(A^*_\xi X, W\right) = B(X,W), \qquad g\left(A^*_\xi X, N\right) = 0,\\ B(X,\xi) = 0, \qquad g(A_N X, N) = 0.\end{gathered} \tag{2.15}$$

Moreover, from the first and third equations of (2.15) we have

$$A^*_\xi \xi = 0. \tag{2.16}$$

Now, we will denote $\overline{R}$ and R the curvature tensors of the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ and the induced connection ∇ on M. Then the Gauss equation of M is given by

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ A_{h(X,Z)}Y - A_{h(Y,Z)}X\\ &+ (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \quad \forall X,Y,Z\in\Gamma(TM),\end{aligned} \tag{2.17}$$

where $(\nabla_X h)(Y,Z) = \nabla^t_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. Then the Gauss-Codazzi equations of a lightlike hypersurface are given by

$$\begin{aligned}\overline{g}\left(\overline{R}(X,Y)Z, PW\right) &= g(R(X,Y)Z, PW)\\ &\quad + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW),\\ \overline{g}\left(\overline{R}(X,Y)Z, \xi\right) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y),\\ \overline{g}\left(\overline{R}(X,Y)Z, N\right) &= g(R(X,Y)Z, N),\\ \overline{g}\left(\overline{R}(X,Y)\xi, N\right) &= g(R(X,Y)\xi, N)\\ &= C\left(Y, A^*_\xi X\right) - C\left(X, A^*_\xi Y\right) - 2d\tau(X,Y),\end{aligned} \tag{2.18}$$

for any $X,Y,Z,W\in\Gamma(TM), \xi\in\Gamma(\text{Rad } TM)$.

For geometries of lightlike submanifolds, hypersurfaces and curves, we refer to [1–3].

2.1. Product Manifolds

Let $\overline{M}$ be an n-dimensional differentiable manifold with a tensor field F of type (1,1) on $\overline{M}$ such that

$$F^2 = I. \tag{2.19}$$

Then $\overline{M}$ is called an almost product manifold with almost product structure F. If we put

$$\pi = \frac{1}{2}(I + F), \qquad \sigma = \frac{1}{2}(I - F), \tag{2.20}$$

then we have

$$\begin{gathered} \pi + \sigma = I, \qquad \pi^2 = \pi, \qquad \sigma^2 = \sigma, \\ \sigma\pi = \pi\sigma = 0, \qquad F = \pi - \sigma. \end{gathered} \tag{2.21}$$

Thus π and σ define two complementary distributions and F has the eigenvalue of $+1$ or -1. If an almost product manifold $\overline{M}$ admits a semi-Riemannian metric $\overline{g}$ such that

$$\overline{g}(FX, FY) = \overline{g}(X, Y), \tag{2.22}$$

for any vector fields X, Y on $\overline{M}$, then $\overline{M}$ is called a semi-Riemannian almost product manifold. From (2.19) and (2.22), we have

$$\overline{g}(FX, Y) = \overline{g}(X, FY). \tag{2.23}$$

If, for any vector fields X, Y on $\overline{M}$,

$$\overline{\nabla} F = 0, \quad \text{that is } \overline{\nabla}_X FY = F\overline{\nabla}_X Y, \tag{2.24}$$

then $\overline{M}$ is called a semi-Riemannian product manifold, where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$.

3. Lightlike Hypersurfaces of Semi-Riemannian Product Manifolds

Let M be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$. For any $X \in \Gamma(TM)$ we can write

$$FX = fX + w(X)N, \tag{3.1}$$

where f is a (1,1) tensor field and w is a 1-form on M given by $w(X) = \overline{g}(FX, \xi) = \overline{g}(X, F\xi)$.

Definition 3.1. Let M be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$:

(i) if F Rad $TM \subset S(TM)$ and F ltr$(TM) \subset S(TM)$ then we say that M is a screen semi-invariant lightlike hypersurface;

(ii) if $FS(TM) = S(TM)$ then we say that M is a screen invariant lightlike hypersurface;

(iii) if F RadTM = ltr (TM) then we say that M is a radical anti-invariant lightlike hypersurface.

We note that a radical anti-invariant lightlike hypersurface is a screen invariant lightlike hypersurface.

Remark 3.2. We recall that there are some lightlike hypersurfaces of a semi-Riemannian product manifold which differ from the above definition, that is, this definition does not cover all lightlike hypersurfaces of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$. In this paper we will study the hypersurfaces determined above.

Now, let M be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold. If we set $\mathbb{D}_1 = F$ Rad $TM, \mathbb{D}_2 = F$ ltr(TM) then we can write

$$S(TM) = \mathbb{D} \perp \{\mathbb{D}_1 \oplus \mathbb{D}_2\}, \tag{3.2}$$

where $\mathbb{D}$ is a $(m-2)$-dimensional distribution. Hence we have the following decomposition:

$$\begin{aligned} TM &= \mathbb{D} \perp \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \perp \text{Rad } TM, \\ T\overline{M} &= \mathbb{D} \perp \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \perp \{\text{Rad } TM \oplus \text{ltr}(TM)\}. \end{aligned} \tag{3.3}$$

Proposition 3.3. *The distribution $\mathbb{D}$ is an invariant distribution with respect to F.*

Proof. For any $X \in \Gamma(\mathbb{D})$ and $U \in \Gamma(\mathbb{D}_1), V \in \Gamma(\mathbb{D}_2)$ we obtain

$$\begin{aligned} g(FX, U) &= g(X, FU) = 0, \\ g(FX, V) &= g(X, FV) = 0. \end{aligned} \tag{3.4}$$

Thus there are no components of FX in $\mathbb{D}_1$ and $\mathbb{D}_2$. Furthermore, we have

$$\begin{aligned} g(FX, \xi) &= g(X, F\xi) = 0, \\ g(FX, N) &= g(X, FN) = 0. \end{aligned} \tag{3.5}$$

Proof is completed. □

If we set $\overline{\mathbb{D}} = \mathbb{D} \perp \operatorname{Rad} TM \perp F \operatorname{Rad} TM$, we can write

$$TM = \overline{\mathbb{D}} \oplus \mathbb{D}_2. \tag{3.6}$$

From the above proposition we have the following corollary.

Corollary 3.4. *The distribution $\overline{\mathbb{D}}$ is invariant with respect to F.*

Example 3.5. Let $(\overline{M} = R^5_2, \overline{g})$ be a 5-dimensional semi-Euclidean space with signature $(-,+,-,+,+)$ and (x,y,z,s,t) be the standard coordinate system of R^5_2. If we set $F(x,y,z,s,t) = (x,y,-z,-s,-t)$, then $F^2 = I$ and F is a product structure on R^5_2. Consider a hypersurface M in $\overline{M}$ by the equation:

$$t = x + y + z. \tag{3.7}$$

Then $TM = \operatorname{Span}\{U_1, U_2, U_3, U_4\}$, where

$$U_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \qquad U_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \qquad U_3 = \frac{\partial}{\partial z} + \frac{\partial}{\partial t}, \qquad U_4 = \frac{\partial}{\partial s}. \tag{3.8}$$

It is easy to check that M is a lightlike hypersurface and

$$TM^{\perp} = \operatorname{Span}\{\xi = U_1 - U_2 + U_3\}. \tag{3.9}$$

Then take a lightlike transversal vector bundle as follow:

$$\operatorname{ltr}(TM) = \operatorname{Span}\left\{ N = -\frac{1}{4}\left\{ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right\} \right\}. \tag{3.10}$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = U_4, W_2 = U_1 - U_2 - U_3, W_3 = U_1 + U_2 - U_3\}. \tag{3.11}$$

If we set $\mathbb{D} = \operatorname{Span}\{W_1\}$, $\mathbb{D}_1 = \operatorname{Span}\{W_2\}$ and $\mathbb{D}_2 = \operatorname{Span}\{W_3\}$, then it can be easily checked that M is a screen semi-invariant lightlike hypersurface of $\overline{M}$.

Example 3.6. Let (x,y,z,t) be the standard coordinate system of R^4 and $ds^2 = -dx^2 - dy^2 + dz^2 + dt^2$ be a semi-Riemannian metric on R^4 with 2-index. Let F be a product structure on R^4 given by $F(x,y,z,t)=(z,t,x,y)$. We consider the hypersurface M given by $t=x+(1/2)(y+z)^2$ [1]. One can easily see that M is a lightlike hypersurface and

$$\begin{aligned}\text{Rad}\,TM &= \text{Span}\left\{\xi=\frac{\partial}{\partial x}+(y+z)\frac{\partial}{\partial y}-(y+z)\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right\},\\ \text{ltr}(TM) &= \text{Span}\left\{N=-\frac{1}{2\left(1+(y+z)^2\right)}\left(\frac{\partial}{\partial x}+(y+z)\frac{\partial}{\partial y}+(y+z)\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)\right\},\\ S(TM) &= \text{Span}\left\{W_1=-(y+z)\frac{\partial}{\partial x}+\frac{\partial}{\partial y},W_2=\frac{\partial}{\partial z}+(y+z)\frac{\partial}{\partial t}\right\}.\end{aligned} \tag{3.12}$$

We can easily check that

$$F\xi=W_1+W_2,\qquad FN=\frac{1}{2\left(1+(y+z)^2\right)}\{W_1-W_2\}. \tag{3.13}$$

Thus M is a screen semi-invariant lightlike hypersurface with $\mathbb{D}=\{0\}$, $\mathbb{D}_1=\text{Span}\{F\xi\}$ and $\mathbb{D}_2=\text{Span}\{FN\}$.

Example 3.7. Let $(R_2^4,\overline{g})$ be a 4-dimensional semi-Euclidean space with signature $(-,-,+,+)$ and (x_1,x_2,x_3,x_4) be the standard coordinate system of R_2^4. Consider a Monge hypersurface M of R_2^4 given by

$$x_4=Ax_1+Bx_2+Cx_3,\qquad A^2+B^2-C^2=1,\quad A,B,C\in R. \tag{3.14}$$

Then the tangent bundle TM of the hypersurface M is spanned by

$$\left\{U_1=\frac{\partial}{\partial x_1}+A\frac{\partial}{\partial x_4},U_2=\frac{\partial}{\partial x_2}+B\frac{\partial}{\partial x_4},U_3=\frac{\partial}{\partial x_3}+C\frac{\partial}{\partial x_4}\right\}. \tag{3.15}$$

It is easy to check that M is a lightlike hypersurface (p.196, Ex.1, [3]) whose radical distribution Rad TM is spanned by

$$\xi=AU_1+BU_2-CU_3=A\frac{\partial}{\partial x_1}+B\frac{\partial}{\partial x_2}-C\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}. \tag{3.16}$$

Furthermore, the lightlike transversal vector bundle is given by

$$\text{ltr}(TM)=\text{Span}\left\{N=-\frac{1}{2(C^2+1)}\left(A\frac{\partial}{\partial x_1}+B\frac{\partial}{\partial x_2}+C\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_4}\right)\right\}. \tag{3.17}$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\left\{W_1=\frac{1}{A^2+B^2}\left(B\frac{\partial}{\partial x_1}-A\frac{\partial}{\partial x_2}\right),W_2=\frac{1}{A^2+B^2}\left(\frac{\partial}{\partial x_3}+C\frac{\partial}{\partial x_4}\right)\right\}. \tag{3.18}$$

If we define a mapping F by $F(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4)$ then $F^2 = I$ and F is a product structure on R_2^4. One can easily check that $FS(TM) = S(TM)$ and F Rad $TM = \mathrm{ltr}(TM)$. Thus M is a radical anti-invariant lightlike hypersurface of R_2^4. Furthermore, this lightlike hypersurface is a screen invariant lightlike hypersurface.

Theorem 3.8. *Let $(\overline{M}, \overline{g})$ be a semi-Riemannian product manifold and M be a screen semi-invariant lightlike hypersurface of $\overline{M}$. Then the following assertions are equivalent.*

(i) *The distribution $\overline{\mathbb{D}}$ is integrable with respect to the induced connection ∇ of M.*

(ii) *$B(X, fY) = B(Y, fX)$, for any $X, Y \in \Gamma(\overline{\mathbb{D}})$.*

(iii) *$g(A_\xi^* X, PfY) = g(A_\xi^* Y, PfX)$, for any $X, Y \in \Gamma(\overline{\mathbb{D}})$.*

Proof. For any $X, Y \in \Gamma(\overline{\mathbb{D}})$, from (2.9), (2.24), and (3.1), we obtain

$$f\nabla_X Y + w(\nabla_X Y)N + B(X, Y)FN = \nabla_X fY + B(X, fY)N. \tag{3.19}$$

Interchanging role of X and Y we have

$$f\nabla_Y X + w(\nabla_Y X)N + B(Y, X)FN = \nabla_Y fX + B(Y, fX)N. \tag{3.20}$$

From (3.19), (3.20) we get

$$w([X, Y]) = B(X, fY) - B(Y, fX) \tag{3.21}$$

and this is (i) ⇔ (ii). From the first equation of (2.15), we conclude (ii) ⇔ (iii). Thus we have our assertion. □

From the decomposition (3.6), we can give the following definition.

Definition 3.9. Let M be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. If $B(X, Y) = 0$, for any $X \in \Gamma(\overline{\mathbb{D}}), Y \in \Gamma(\mathbb{D}_2)$, then we say that M is a mixed geodesic lightlike hypersurface.

Theorem 3.10. *Let $(\overline{M}, \overline{g})$ be a semi-Riemannian product manifold and M be a screen semi-invariant lightlike hypersurface of $\overline{M}$. Then the following assertions are equivalent.*

(i) *M is mixed geodesic.*

(ii) *There is no $\mathbb{D}_2$-component of A_N.*

(iii) *There is no $\mathbb{D}_1$-component of A_ξ^*.*

Proof. Suppose that M is mixed geodesic screen semi-invariant lightlike hypersurface of $\overline{M}$ with respect to the Levi-Civita connection $\overline{\nabla}$. From (2.24), (2.9), (2.10), and (3.1), we obtain

$$\nabla_X FN + B(X, FN)N = -fA_N X + \tau(X)FN - w(A_N X)N, \tag{3.22}$$

for any $X \in \Gamma(\overline{\mathbb{D}})$. If we take tangential and transversal parts of this last equation we have

$$\nabla_X FN = -f A_N X + \tau(X) FN,$$
$$B(X, FN) = -w(A_N X). \tag{3.23}$$

Furthermore, since $w(A_N X) = g(A_N X, F\xi)$, we get (i) $\Leftrightarrow$ (ii). Since $\overline{g}(FN, \xi) = \overline{g}(N, F\xi) = 0$, we obtain

$$g(A_N X, F\xi) = -g\left(A^*_\xi X, FN\right). \tag{3.24}$$

This is (ii) $\Leftrightarrow$ (iii). □

From the decomposition (3.6), we have the following theorem.

Theorem 3.11. *Let M be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then M is a locally product manifold according to the decomposition* (3.6) *if and only if f is parallel with respect to induced connection ∇, that is $\nabla f = 0$.*

Proof. Let M be a locally product manifold. Then the leaves of distributions $\overline{\mathbb{D}}$ and $\mathbb{D}_2$ are both totally geodesic in M. Since the distribution $\overline{\mathbb{D}}$ is invariant with respect to F then, for any $Y \in \Gamma(\overline{\mathbb{D}})$, $FY \in \Gamma(\overline{\mathbb{D}})$. Thus $\nabla_X Y$ and $\nabla_X fY$ belong to $\Gamma(\overline{\mathbb{D}})$, for any $X \in \Gamma(TM)$. From the Gauss formula, we obtain

$$\nabla_X fY + B(X, fY)N = f\nabla_X Y + w(\nabla_X Y)N + B(X, Y)FN. \tag{3.25}$$

Comparing the tangential and normal parts with respect to $\overline{\mathbb{D}}$ of (3.25), we have

$$\nabla_X fY = f\nabla_X Y, \quad \text{that is } (\nabla_X f)Y = 0, \tag{3.26}$$

$$B(X, Y) = 0. \tag{3.27}$$

Since $fZ = 0$, for any $Z \in \Gamma(\mathbb{D}_2)$, we get $\nabla_X fZ = 0$ and $f\nabla_X Z = 0$, that is $(\nabla_X f)Z = 0$. Thus we have $\nabla f = 0$ on M.

Conversely, we assume that $\nabla f = 0$ on M. Then we have $\nabla_X fY = f\nabla_X Y$, for any $X, Y \in \Gamma(\overline{\mathbb{D}})$ and $\nabla_U fW = f\nabla_U W = 0$, for any $U, W \in \Gamma(\mathbb{D}_2)$. Thus it follows that $\nabla_X fY \in \Gamma(\overline{\mathbb{D}})$ and $\nabla_U W \in \Gamma(\mathbb{D}_2)$. Hence, the leaves of the distributions $\overline{\mathbb{D}}$ and $\mathbb{D}_2$ are totally geodesic in M. □

From Theorem 3.11 and (3.27) we have the following corollary.

Corollary 3.12. *Let M be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. If M has a local product structure, then it is a mixed geodesic lightlike hypersurface.*

Let M be a radical anti-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then we have the following decomposition:

$$T\overline{M} = S(TM) \perp \{\text{Rad } TM \oplus F \text{ Rad } TM\}. \tag{3.28}$$

Theorem 3.13. *Let M be a radical anti-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then the screen distribution $S(TM)$ of M is an integrable distribution if and only if $B(X, FY) = B(Y, FX)$.*

Proof. If a vector field X on M belongs to $S(TM)$ if and only if $\eta(X) = 0$. Since M is a radical anti-invariant lightlike hypersurface, for any $X \in \Gamma(S(TM))$, $FX \in \Gamma(S(TM))$. For any $X, Y \in \Gamma(S(TM))$, we can write

$$\overline{\nabla}_X FY = \nabla_X FY + B(X, FY)N. \tag{3.29}$$

In this last equation interchanging role of X and Y, we obtain

$$F[X, Y] = \nabla_X FY - \nabla_Y FX + (B(X, FY) - B(Y, FX))N. \tag{3.30}$$

Since $\eta([X, Y]) = \overline{g}([X, Y], N) = \overline{g}(F[X, Y], FN)$, we get

$$\eta([X, Y]) = (B(X, FY) - B(Y, FX))\overline{g}(N, FN). \tag{3.31}$$

Since $\overline{g}(N, FN) \neq 0$, $\eta([X, Y]) = 0$ if and only if $B(X, FY) = B(Y, FX)$. This is our assertion. □

4. Quarter-Symmetric Nonmetric Connections

Let $(\overline{M}, \overline{g}, F)$ be a semi-Riemannian product manifold and $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. If we set

$$\overline{D}_X Y = \overline{\nabla}_X Y + u(Y)FX, \tag{4.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, then $\overline{D}$ is a linear connection on $\overline{M}$, where u is a 1-form on $\overline{M}$ with U as associated vector field, that is

$$u(X) = \overline{g}(X, U). \tag{4.2}$$

The torsion tensor of $\overline{D}$ on $\overline{M}$ denoted by $\overline{T}$. Then we obtain

$$\overline{T}(X, Y) = u(Y)FX - u(X)FY, \tag{4.3}$$

$$\left(\overline{D}_X \overline{g}\right)(Y, Z) = -u(Y)\overline{g}(FX, Z) - u(Z)\overline{g}(FX, Y), \tag{4.4}$$

for any $X, Y \in \Gamma(T\overline{M})$. Thus $\overline{D}$ is a quarter-symmetric nonmetric connection on $\overline{M}$. From (2.24) and (4.1) we have

$$\left(\overline{D}_X F\right)Y = u(FY)FX - u(Y)X. \tag{4.5}$$

Replacing X by FX and Y by FY in (4.5) and using (2.19) we obtain

$$\left(\overline{D}_{FX} F\right)FY = u(Y)X - u(FY)FX. \tag{4.6}$$

Thus we have

$$\left(\overline{D}_X F\right)Y + \left(\overline{D}_{FX} F\right)FY = 0. \tag{4.7}$$

If we set

$$'F(X, Y) = \overline{g}(FX, Y), \tag{4.8}$$

for any $X, Y \in \Gamma(T\overline{M})$, from (4.1) we get

$$\left(\overline{D}_X {}'F\right)(Y, Z) = \left(\overline{\nabla}_X {}'F\right)(Y, Z) - u(Y)\overline{g}(X, Z) - u(Z)\overline{g}(X, Y). \tag{4.9}$$

From (4.1) the curvature tensor $\overline{R}^D$ of the quarter-symmetric nonmetric connection $\overline{D}$ is given by

$$\overline{R}^D(X, Y)Z = \overline{R}(X, Y)Z + \overline{\lambda}(X, Z)FY - \overline{\lambda}(Y, Z)FX, \tag{4.10}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$, where $\overline{\lambda}$ is a $(0, 2)$-tensor given by $\overline{\lambda}(X, Z) = (\overline{\nabla}_X u)(Z) - u(Z)u(FX)$. If we set $\overline{R}^D(X, Y, Z, W) = \overline{g}(\overline{R}^D(X, Y)Z, W)$, then, from (4.10), we obtain

$$\overline{R}^D(X, Y, Z, W) = -\overline{R}^D(Y, X, Z, W). \tag{4.11}$$

We note that the Riemannian curvature tensor $\overline{R}^D$ of $\overline{D}$ does not satisfy the other curvature-like properties. But, from (4.10), we have

$$\begin{aligned} \overline{R}^D(X, Y)Z + \overline{R}^D(Y, Z)X + \overline{R}^D(Z, X)Y = &\left(\overline{\lambda}(Z, Y) - \overline{\lambda}(Y, Z)\right)FX \\ &+ \left(\overline{\lambda}(X, Z) - \overline{\lambda}(Z, X)\right)FY \\ &+ \left(\overline{\lambda}(Y, X) - \overline{\lambda}(X, Y)\right)FZ. \end{aligned} \tag{4.12}$$

Thus we have the following proposition.

Proposition 4.1. *Let M be a lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then the first Bianchi identity of the quarter-symmetric nonmetric connection $\overline{D}$ on M is provided if and only if $\overline{\lambda}$ is symmetric.*

Let M be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$ with quarter-symmetric nonmetric connection $\overline{D}$. Then the Gauss and Weingarten formulas with respect to $\overline{D}$ are given by, respectively,

$$\overline{D}_X Y = D_X Y + \overline{B}(X, Y) N \tag{4.13}$$

$$\overline{D}_X N = -\overline{A}_N X + \overline{\tau}(X) N \tag{4.14}$$

for any $X, Y \in \Gamma(TM)$, where $D_X Y$, $\overline{A}_N X \in \Gamma(TM)$, $\overline{B}(X, Y) = \overline{g}(\overline{D}_X Y, \xi)$, $\overline{\tau}(X) = \overline{g}(\overline{D}_X N, \xi)$. Here, D, $\overline{B}$ and $\overline{A}_N$ are called the induced connection on M, the second fundamental form, and the Weingarten mapping with respect to $\overline{D}$. From (2.9), (2.10), (3.1), (4.1), (4.13), and (4.14) we obtain

$$D_X Y = \nabla_X Y + u(Y) f X, \tag{4.15}$$

$$\overline{B}(X, Y) = B(X, Y) + u(Y) w(X), \tag{4.16}$$

$$\begin{aligned} \overline{A}_N X &= A_N X - u(N) f X, \\ \overline{\tau}(X) &= \tau(X) + u(N) w(X), \end{aligned} \tag{4.17}$$

for any X, $Y \in \Gamma(TM)$. From (4.1), (4.4), (4.13), and (4.16) we get

$$\begin{aligned} (D_X g)(Y, Z) = {} & B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ & - u(Y) g(fX, Z) - u(Z) g(fX, Y). \end{aligned} \tag{4.18}$$

On the other hand, the torsion tensor of the induced connection D is

$$T^D(X, Y) = u(Y) f X - u(X) f Y. \tag{4.19}$$

From last two equations we have the following proposition.

Proposition 4.2. *Let M be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$ with quarter-symmetric nonmetric connection $\overline{D}$. Then the induced connection D is a quarter-symmetric nonmetric connection on the lightlike hypersurface M.*

For any $X, Y \in \Gamma(TM)$, we can write

$$\begin{aligned} D_X\ PY &= D^*_X PY + \overline{C}(X, PY)\xi, \\ D_X\ \xi &= -\overline{A}^*_\xi X + \varepsilon(X)\xi, \end{aligned} \tag{4.20}$$

where $D^*_X PY\ \overline{A}^*_\xi X \in \Gamma(S(TM))$, $\overline{C}(X,PY) = \overline{g}(D_X PY, N)$, and $\varepsilon(X) = \overline{g}(D_X\xi, N)$. From (2.14), (16), and (4.15), we obtain

$$\overline{C}(X,PY) = C(X,PY) + u(PY)\eta(fX), \tag{4.21}$$

$$\overline{A}^*_\xi X = A^*_\xi X - u(\xi)PfX, \qquad \varepsilon(X) = -\tau(X) + u(\xi)\eta(fX). \tag{4.22}$$

Using (2.15), (4.16) and (4.22) we obtain

$$\begin{aligned}\overline{B}(X,PY) = g\left(\overline{A}^*_\xi X, PY\right) + u(PY)w(X) \\ + u(\xi)\overline{g}(FX,PY),\end{aligned} \tag{4.23}$$

for any $X, Y \in \Gamma(TM)$.

Now, we consider a screen semi-invariant lightlike hypersurface M of a semi-Rieamannian product manifold $\overline{M}$ with respect to the quarter symmetric connection $\overline{D}$ given by (4.1). Since $w(X) = g(FX,\xi)$, for any $X \in \Gamma(\mathbb{D}), w(X) = 0$. Thus we have the following propositions.

Proposition 4.3. *Let M be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M},\overline{g})$ with quarter-symmetric nonmetric connection. The second fundamental form $\overline{B}$ of quarter-symmetric nonmetric connection $\overline{D}$ is degenerate.*

Proposition 4.4. *Let $(\overline{M},\overline{g})$ be a semi-Riemannian product manifold and M be a screen semi-invariant lightlike hypersurfaces of $\overline{M}$. If M is $\mathbb{D}$ totally geodesic with respect to $\overline{\nabla}$, then M is $\mathbb{D}$ totally geodesic with respect to quarter-symmetric nonmetric connection.*

Theorem 4.5. *Let $(\overline{M},\overline{g})$ be a semi-Riemannian product manifold and M be a screen semi-invariant lightlike hypersurfaces of $\overline{M}$. Then the following assertions are equivalent.*

(i) *The distribution $\overline{\mathbb{D}}$ is integrable with respect to the quarter symmetric nonmetric connection D.*

(ii) $\overline{B}(X,fY) = \overline{B}(Y,fX)$, *for any* $X, Y \in \Gamma(\overline{\mathbb{D}})$.

(iii) $g(\overline{A}^*_\xi X, PfY) = g(\overline{A}^*_\xi Y, PfX)$, *for any* $X, Y \in \Gamma(\overline{\mathbb{D}})$.

The proof of this theorem is similar to the proof of the Theorem 3.8.

From (4.23), for any $X \in \Gamma(\mathbb{D})$ and $Y \in \Gamma(\mathbb{D}_2)$, we have $\overline{B}(X,PY) = g(\overline{A}^*_\xi X, PY)$. If we set $\mathbb{D}' = \mathbb{D} \perp \mathbb{D}_2$, then, from Theorem 3.10, we have the following corollary.

Corollary 4.6. *Let $(\overline{M},\overline{g})$ be a semi-Riemannian product manifold and M be a screen semi-invariant lightlike hypersurface of $\overline{M}$. Then the distribution $\mathbb{D}'$ is a mixed geodesic foliation defined with respect to quarter symmetric nonmetric connection if and only if there is no $\mathbb{D}_1$ component of $\overline{A}^*_\xi$.*

From (4.15), we obtain

$$
\begin{aligned}
R^D(X,Y)Z = R(X,Y)Z &+ u(Z)\{(\nabla_X f)Y - (\nabla_Y f)X\} \\
&+ \lambda(X,Z)fY - \lambda(Y,Z)fX,
\end{aligned} \tag{4.24}
$$

where λ is a $(0,2)$ tensor on M given by $\lambda(X,Z) = (\nabla_X u)(Z) - u(Z)u(fX)$.

From (4.24), we have the following proposition which is similar to the Proposition 4.1.

Proposition 4.7. *Let M be a lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. One assumes that f is parallel on M. Then the first Bianchi identity of the quarter-symmetric nonmetric connection D on M is provided if and only if λ is symmetric.*

Now we will compute Gauss-Codazzi equations of lightlike hypersurfaces with respect to the quarter-symmetric nonmetric connection:

$$
\begin{aligned}
\overline{g}\left(\overline{R}^D(X,Y)Z,PW\right) &= g(R(X,Y)Z,PW) \\
&\quad + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) \\
&\quad + \overline{\lambda}(X,Z)g(fY,PW) - \overline{\lambda}(Y,Z)g(fX,PW), \\
\overline{g}\left(\overline{R}^D(X,Y)Z,\xi\right) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\
&\quad + \overline{\lambda}(X,Z)w(Y) - \overline{\lambda}(Y,Z)w(X), \\
\overline{g}\left(\overline{R}^D(X,Y)Z,N\right) &= g(R(X,Y)Z,N) \\
&\quad + \overline{\lambda}(X,Z)\eta(fY) - \overline{\lambda}(Y,Z)\eta(fX),
\end{aligned} \tag{4.25}
$$

for any X, Y, Z, $W \in \Gamma(TM)$.

Now, let M be a screen semi-invariant lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian product manifold with the quarter-symmetric nonmetric connection $\overline{D}$ such that the tensor field f is parallel on M. We consider the local quasiorthonormal basis $\{E_i, F\xi, FN, \xi, N\}$, $i = 1, \ldots m-2$, of $\overline{M}$ along M, where $\{E_1, \ldots, E_{m-2}\}$ is an orthonormal basis of $\Gamma(\mathbb{D})$. Then, the Ricci tensor of M with respect to D is given by

$$
\begin{aligned}
R^{D(0,2)}(X,Y) = \sum_{i=1}^{m-2}\varepsilon_i g\left(R^D(X,E_i)Y,E_i\right) &+ g\left(R^D(X,F\xi)Y,FN\right) \\
&+ g\left(R^D(X,FN)Y,F\xi\right) + g\left(R^D(X,\xi)Y,N\right).
\end{aligned} \tag{4.26}
$$

From (4.24) we have

$$R^{D(0,2)}(X,Y) = R^{(0,2)}(X,Y) + \sum_{i=1}^{m-2}\varepsilon_i\{\lambda(X,Y)g(fE_i,E_i) - \lambda(E_i,Y)g(fX,E_i)\} - \lambda(F\xi,Y)\eta(X) - \lambda(\xi,Y)\eta(fX), \tag{4.27}$$

where $R^{(0,2)}(X,Y)$ is the Ricci tensor of M. Thus we have the following corollary.

Corollary 4.8. *Let M a screen semi-invariant lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian product manifold with the quarter-symmetric nonmetric connection $\overline{D}$ such that the tensor field f is parallel on M and $R^{(0,2)}(X,Y)$ is symmetric. Then $R^{D(0,2)}$ is symmetric on the distribution $\mathbb{D}$ if and only if λ is symmetric and $\lambda(fX,Y) = \lambda(fY,X)$.*

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