



SOME KOROVKIN TYPE RESULTS VIA POWER SERIES METHOD IN MODULAR SPACES

T. YURDAKADIM

ABSTRACT. In this paper, we obtain a Korovkin type approximation result for a sequence of positive linear operators defined on modular spaces with the use of power series method . We also provide an example which satisfies our theorem.

1. INTRODUCTION

The classical Korovkin theorem states the uniform convergence of a sequence of positive linear operators in $C[a, b]$, the space of all continuous real valued functions defined on $[a, b]$ by providing the convergence only on three test functions $\{1, x, x^2\}$. There are also trigonometric versions of this theorem with the test functions $\{1, \cos x, \sin x\}$ and abstract Korovkin type results have also been studied [13, 17]. These type of results let us to say the convergence with minimum calculations and also have important applications in the polynomial approximation theory, in various areas of functional analysis, in numerical solutions of differential and integral equations [1, 2] . Recently some versions of Korovkin type theorems have been given in modular spaces that include as particular cases L^p , Orlicz and Musielak-Orlicz spaces [8, 19] with the use of more general convergences such as convergences generated by summability methods, statistical, filter convergence [9, 10, 11, 14, 15, 16, 20].

In this paper, we give a Korovkin type theorem in modular spaces by power series method which includes both Abel and Borel methods. We also give an example which satisfies our theorems.

2. NOTATION AND DEFINITIONS

Let us begin with recalling some basic definitions and notations used in the paper.

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Let (p_j) be real sequence with $p_0 > 0$ and $p_1, p_2, p_3, \dots \geq 0$, and such that the corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If, for all $t \in (0, R)$,

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L$$

then we say that $x = (x_j)$ is convergent in the sense of power series method [18, 21]. Power series method includes many well known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and that they are not matrix methods (See [12, 22] for details). In order to see that power series method is more effective than ordinary convergence, let $x = (1, 0, 1, 0, \dots)$, $R = \infty$, $p(t) = e^t$ and for $j \geq 0$, $p_j = \frac{1}{j!}$. Then it is easy to see that

$$\lim_{t \rightarrow \infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{x_j t^j}{j!} = \lim_{t \rightarrow \infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} = \lim_{t \rightarrow \infty} \frac{1}{e^t} \left\{ \frac{e^t + e^{-t}}{2} \right\} = \frac{1}{2}.$$

So the sequence $x = (x_j)$ is convergent to $\frac{1}{2}$ in the sense of power series method but it is not convergent in the ordinary sense. Note that the power series method is regular if and only if

$$\lim_{t \rightarrow R^-} \frac{p_j t^j}{p(t)} = 0, \text{ for each } j \in \mathbb{N}$$

hold [12]. Throughout the paper we assume that power series method is regular.

Let $G = [a, b]$ be a bounded interval of the real line \mathbb{R} provided with the Lebesgue measure. We denote by $X(G)$ the space of all real-valued measurable functions on G with equality almost everywhere, by $C(G)$ the space of all continuous real valued functions on G , and by $C^\infty(G)$ the space of all infinitely differentiable functions on G . A functional $\varrho : X(G) \rightarrow [0, \infty]$ is a modular on $X(G)$ provided that the following conditions hold:

- (i) $\varrho[f] = 0$ if and only if $f = 0$ a.e on G ,
- (ii) $\varrho[-f] = \varrho[f]$ for every $f \in X(G)$,
- (iii) $\varrho[\alpha f + \beta g] \leq \varrho[f] + \varrho[g]$ for every $f, g \in X(G)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ϱ is said to be Q -quasi convex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[\alpha f + \beta g] \leq Q\alpha\varrho[Qf] + Q\beta\varrho[Qg]$$

holds for every $f, g \in X(G)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular if $Q = 1$, then ϱ is called convex.

A modular ϱ is said to be Q -quasi semiconvex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[af] \leq Qa\varrho[Qf]$$

holds for every $f \in X(G)$, $f \geq 0$ and $a \in (0, 1]$. It is clear that every Q -quasi semiconvex modular is Q -quasi convex. A modular ϱ is said to be monotone if $\varrho[f] \leq \varrho[g]$ for all $f, g \in X(G)$ with $|f| \leq |g|$.

We now consider some subspaces of $X(G)$ by means of a modular ϱ as follows

$$L^\varrho(G) := \{f \in X(G) : \lim_{\lambda \rightarrow 0^+} \varrho[\lambda f] = 0\}$$

and

$$E^\varrho(G) := \{f \in L^\varrho(G) : \varrho[\lambda f] < \infty \text{ for all } \lambda > 0\}$$

is called the modular space generated by ϱ and is called the space of the finite elements of $L^\varrho(G)$ respectively. Observe that if ϱ is Q -quasi semiconvex then the space

$$\{f \in X(G) : \varrho[\lambda f] < \infty \text{ for some } \lambda > 0\}$$

coincides with $L^\varrho(G)$. The notions about modulars have been introduced and widely discussed in [4, 5, 6, 7, 8].

Now we define the convergences in the sense of power series method in modular spaces. Let $\{f_j\}$ be a function sequence whose terms belong to $L^\varrho(G)$. Then, $\{f_j\}$ is modularly convergent to a function $f \in L^\varrho(G)$ in the sense of power series method if and only if

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(f_j - f)] = 0 \text{ for some } \lambda_0 > 0.$$

Also, $\{f_j\}$ is strongly convergent to a function $f \in L^\varrho(G)$ in the sense of power series method if and only if

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(f_j - f)] = 0 \text{ for every } \lambda > 0.$$

Recall that $\{f_j\}$ is modularly convergent to a function $f \in L^\varrho(G)$ if and only if

$$\lim_{j \rightarrow \infty} \varrho[\lambda_0(f_j - f)] = 0 \text{ for some } \lambda_0 > 0,$$

also $\{f_j\}$ is strongly convergent to a function $f \in L^\varrho(G)$ if and only if

$$\lim_{j \rightarrow \infty} \varrho[\lambda(f_j - f)] = 0 \text{ for every } \lambda > 0.$$

If there exists a constant $M > 0$ such that

$$\varrho[2u] \leq M\varrho[u]$$

holds for all $u \geq 0$ then it is said to be that ϱ satisfies the Δ_2 -condition. A modular ϱ is said to be

- finite if χ_G , the characteristic function associated with G , belongs to $L^\varrho(G)$,

- absolutely finite if ϱ is finite and for every $\varepsilon > 0$, $\lambda > 0$ there exists $\delta > 0$ such that $\varrho[\lambda\chi_B] < \varepsilon$ for any measurable subset $B \subset G$ with $|B| < \delta$,
- strongly finite if $\chi_G \in E^\varrho(G)$,
- absolutely continuous if there is a positive constant a with the property: for all $f \in X(G)$ with $\varrho[f] < \infty$, the following condition holds: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\varrho[af\chi_B] < \varepsilon$ whenever B is any measurable subset of G with $|B| < \delta$.

Recall that if a modular ϱ is monotone and finite, then we have $C(G) \subset L^\varrho(G)$ [4]. In a similar manner, if ϱ is monotone and strongly finite, then $C(G) \subset E^\varrho(G)$.

3. MODULAR KOROVKIN THEOREM BY POWER SERIES METHOD

Let ϱ be monotone and finite modular on $X(G)$. Assume that D is a set satisfying $C^\infty(G) \subset D \subset L^\varrho(G)$. We can construct such a subset D since ϱ is monotone and finite. Assume further that $T := \{T_j\}$ is a sequence of positive linear operators from D into $X(G)$ for which there exists a subset $X_T \subset D$ containing $C^\infty(G)$ such that the inequality

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j h)] \leq P \varrho(\lambda h) \quad (3.1)$$

holds for every $h \in X_T$, $\lambda > 0$ and for an absolute positive constant P . Throughout the paper we use the test functions defined by $e_i(x) = x^i$, $i = 0, 1, 2, \dots$

Theorem 1. *Let ϱ be a strongly finite, monotone, absolutely continuous and Q -quasi semiconvex modular on $X(G)$. Let T_j , $j \in \mathbb{N}$, be a sequence of positive linear operators from D into $X(G)$ satisfying (3.1). If*

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j e_i - e_i)] = 0,$$

for every $\lambda > 0$ and $i = 0, 1, 2$, then for every $f \in L^\varrho(G)$ such that $f - g \in X_T$ for every $g \in C^\infty(G)$

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] = 0,$$

for some $\gamma > 0$.

Proof. Let $g \in C(G)$ and first we show that

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\mu(T_j g - g)] = 0, \text{ for every } \mu > 0. \quad (3.2)$$

Since g is uniformly continuous on G then there exists a constant $M > 0$ such that $|g(x)| \leq M$ for every $x \in G$. Given $\varepsilon > 0$, we can choose $\delta > 0$ such that $|y - x| < \delta$

implies $|g(y) - g(x)| < \varepsilon$ where $x, y \in G$. One can see that for all $x, y \in G$

$$|g(y) - g(x)| < \varepsilon + \frac{2M}{\delta^2}(y - x)^2.$$

Since $\{T_j\}$ is a sequence of positive linear operators, we get

$$\begin{aligned} |T_j(g; x) - g(x)| &= |T_j(g(\cdot) - g(x); x) + g(x)(T_j(e_0(\cdot); x) - e_0(x))| \\ &\leq T_j(|g(\cdot) - g(x)|; x) + |g(x)||T_j(e_0(\cdot); x) - e_0(x)| \\ &\leq T_j(\varepsilon + \frac{2M}{\delta^2}(\cdot - x)^2; x) + M|T_j(e_0(\cdot); x) - e_0(x)| \\ &\leq \varepsilon T_j(e_0(\cdot); x) + \frac{2M}{\delta^2}T_j((\cdot - x)^2; x) + M|T_j(e_0(\cdot); x) - e_0(x)| \\ &\leq \varepsilon + (\varepsilon + M)|T_j(e_0(\cdot); x) - e_0(x)| \\ &\quad + \frac{2M}{\delta^2}|T_j(e_2(\cdot); x) - 2e_1(x)T_j(e_1(\cdot); x) + e_2(x)T_j(e_0(\cdot); x)| \\ &\leq \varepsilon + (\varepsilon + M)|T_j(e_0(\cdot); x) - e_0(x)| + \frac{2M}{\delta^2}|T_j(e_2(\cdot); x) - e_2(x)| \\ &\quad + \frac{4M|e_1(x)|}{\delta^2}|T_j(e_1(\cdot); x) - e_1(x)| + \frac{2Me_2(x)}{\delta^2}|T_j(e_0(\cdot); x) - e_0(x)| \\ &\leq \varepsilon + (\varepsilon + M + \frac{2Mr^2}{\delta^2})|T_j(e_0(\cdot); x) - e_0(x)| \\ &\quad + \frac{4Mr}{\delta^2}|T_j(e_1(\cdot); x) - e_1(x)| + \frac{2M}{\delta^2}|T_j(e_2(\cdot); x) - e_2(x)| \end{aligned}$$

where $r := \max\{|a|, |b|\}$. So the last inequality gives for any $\mu > 0$ that

$$\begin{aligned} \mu|T_j(g; x) - g(x)| &\leq \mu\varepsilon + \mu K|T_j(e_0(\cdot); x) - e_0(x)| + \mu K|T_j(e_1(\cdot); x) - e_1(x)| \\ &\quad + \mu K|T_j(e_2(\cdot); x) - e_2(x)| \end{aligned}$$

where $K := \max\{\varepsilon + M + \frac{2Mr^2}{\delta^2}, \frac{4Mr}{\delta^2}, \frac{2M}{\delta^2}\}$. By applying the modular ϱ in the both sides of the above inequality, since ϱ is monotone, we have

$$\varrho[\mu(T_j(g; \cdot) - g(\cdot))] \leq \varrho[\mu\varepsilon + \mu K|T_j e_0 - e_0| + \mu K|T_j e_1 - e_1| + \mu K|T_j e_2 - e_2|].$$

So we may write that

$$\varrho[\mu(T_j(g; \cdot) - g(\cdot))] \leq \varrho[4\mu\varepsilon] + \varrho[4\mu K(T_j e_0 - e_0)] + \varrho[4\mu K(T_j e_1 - e_1)] + \varrho[4\mu K(T_j e_2 - e_2)].$$

Since ϱ is Q -quasi semiconvex and strongly finite, we have

$$\begin{aligned} \varrho[\mu(T_j(g; \cdot) - g(\cdot))] &\leq Q\varepsilon\varrho[4\mu Q] + \varrho[4\mu K(T_j e_0 - e_0)] + \varrho[4\mu K(T_j e_1 - e_1)] \\ &\quad + \varrho[4\mu K(T_j e_2 - e_2)] \end{aligned}$$

without loss of generality where $0 < \varepsilon \leq 1$. Hence

$$\begin{aligned} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\mu(T_j(g; \cdot) - g(\cdot))] &\leq Q\varepsilon\varrho[4\mu Q] + \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[4\mu K(T_j e_0 - e_0)] \\ &+ \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[4\mu K(T_j e_1 - e_1)] + \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[4\mu K(T_j e_2 - e_2)] \end{aligned}$$

and taking limit superior as $t \rightarrow R^-$ in the both sides, by using hypothesis, we get

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\mu(T_j g - g)] = 0$$

which proves our claim. Now let $f \in L^q(G)$ satisfying $f - g \in X_T$ for every $g \in C^\infty(G)$. Since $|G| < \infty$ and ϱ is strongly finite and absolutely continuous, it is known that ϱ is also absolutely finite on $X(G)$ (see [3]). Using the properties of ϱ and it is also known from [8] that the space $C^\infty(G)$ is modularly dense in $L^q(G)$, i.e., there exists a sequence $\{g_k\} \subset C^\infty(G)$ such that

$$\lim_k \varrho[3\lambda_0(g_k - f)] = 0 \text{ for some } \lambda_0 > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\varrho[3\lambda_0(g_k - f)] < \varepsilon \text{ for every } k \geq k_0.$$

On the other hand, by linearity and positivity of the operators T_j we may write that

$$\lambda_0 |T_j f - f| \leq \lambda_0 |T_j(f - g_{k_0})| + \lambda_0 |T_j g_{k_0} - g_{k_0}| + \lambda_0 |(g_{k_0} - f)|.$$

Applying the modular ϱ in the both sides of the above inequality, since ϱ is monotone

$$\varrho[\lambda_0(T_j f - f)] \leq \varrho[3\lambda_0(T_j(f - g_{k_0}))] + \varrho[3\lambda_0(T_j g_{k_0} - g_{k_0})] + \varrho[3\lambda_0((g_{k_0} - f))].$$

Then it follows from the above inequalities that

$$\varrho[\lambda_0(T_j f - f)] \leq \varrho[3\lambda_0(T_j(f - g_{k_0}))] + \varrho[3\lambda_0(T_j g_{k_0} - g_{k_0})] + \varepsilon.$$

Hence, using the facts that $g_{k_0} \in C^\infty(G)$ and $f - g_{k_0} \in X_T$, we have

$$\begin{aligned} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)] &\leq \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda_0(T_j(f - g_{k_0}))] \\ &+ \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda_0(T_j g_{k_0} - g_{k_0})] + \varepsilon. \end{aligned}$$

Taking limit superior as $t \rightarrow R^-$ in both sides, we obtain that

$$\begin{aligned} \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)] &\leq \varepsilon + P \varrho[3\lambda_0(f - g_{k_0})] \\ &+ \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda_0(T_j g_{k_0} - g_{k_0})] \end{aligned} \quad (3.3)$$

which gives

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)] \leq \varepsilon + \varepsilon P + \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda_0(T_j g_{k_0} - g_{k_0})].$$

By (3.2), we get

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda_0(T_j g_{k_0} - g_{k_0})] = 0$$

and this implies

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)] \leq \varepsilon + \varepsilon P.$$

Since ε is arbitrary positive real number, we have

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)] = 0$$

and also $\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)]$ is nonnegative then

$$\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0(T_j f - f)] = 0.$$

This completes the proof. \square

If the modular ϱ satisfies the Δ_2 -condition, then one can get the following result from the above theorem.

Theorem 2. *Let ϱ and $T = \{T_j\}$ be as in the above theorem. If ϱ satisfies the Δ_2 -condition, then the followings are equivalent:*

- $\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j e_i - e_i)] = 0$, for every $\lambda > 0$ and $i = 0, 1, 2$.
- $\lim_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] = 0$, for every $\lambda > 0$ then every $f \in L^e(G)$ such that $f - g \in X_T$, for every $g \in C^\infty(G)$.

4. CONCLUDING REMARKS

Take $G = [0, 1]$ and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0, \varphi(u) > 0$ for $u > 0$ and $\lim_{u \rightarrow +\infty} \varphi(u) = \infty$.

Here, consider the functional ρ^φ on $X(G)$ defined by

$$\rho^\varphi(f) := \int_0^1 \varphi(|f(x)|) dx, \text{ for } f \in X(G).$$

In this case, ρ^φ is a convex modular on $X(G)$ (see [4]). Consider the Orlicz space generated by φ as follows

$$L_\varphi^\rho(G) := \{f \in L^0(G) : \rho^\varphi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

Then, consider the following classical Bernstein-Kantorovich operator $\mathcal{U} := \{U_j\}$ on the space $L_\varphi^\rho(G)$ (see [4]) which is defined by

$$U_j(f; x) := \sum_{k=0}^j \binom{j}{k} x^k (1-x)^{j-k} (j+1) \int_{\frac{k}{j+1}}^{\frac{k+1}{j+1}} f(t) dt; \quad x \in G.$$

Observe that the operators U_j map the Orlicz space $L_\varphi^\rho(G)$ into itself. Moreover, it is also known that the property $\limsup_{j \rightarrow \infty} \varrho(\lambda(T_j h)) \leq P \varrho(\lambda h)$ is satisfied with the

choice of $X_U := L_\varphi^\rho(G)$ and for every function $f \in L_\varphi^\rho(G)$ such that $f - g \in X_U$ for every $g \in C^\infty(G)$, $\{U_j f\}$ is modularly convergent to f . Using the operators $\{U_j f\}$ define the sequence of positive linear operators $V := V_j$ on $L_\varphi^\rho(G)$ as follows:

$$V_j(f; x) = (1 + s_j) U_j(f; x), \text{ for } f \in L_\varphi^\rho(G), \quad x \in [0, 1] \text{ and } j \in \mathbb{N}, \quad (4.1)$$

where $\{s_j\}$ is a sequence of zeros and ones which is not convergent but convergent to 0 in the sense of power series method. By Lemma 5.1 of [4], for every $h \in X_V := L_\varphi^\rho(G)$, all $\lambda > 0$ and for an absolute positive constant P , we get

$$\begin{aligned} \rho^\varphi(\lambda V_j h) &= \rho^\varphi[\lambda(1 + s_j) U_j h] \leq \rho^\varphi(2\lambda U_j h) + \rho^\varphi(2\lambda s_j U_j h) \\ &= \rho^\varphi(2\lambda U_j h) + s_j \rho^\varphi(2\lambda U_j h) \\ &= (1 + s_j) \rho^\varphi(2\lambda U_j h) \\ &\leq (1 + s_j) P \rho^\varphi(2\lambda h). \end{aligned}$$

Then, we get

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^\varphi(\lambda V_j h) \leq P \rho^\varphi(2\lambda h).$$

Now, we show that conditions in the Theorem 1 holds. First note that

$$\begin{aligned}
V_j(e_0; x) &= 1 + s_j \\
V_j(e_1; x) &= (1 + s_j) \left\{ \frac{jx}{j+1} + \frac{1}{2(j+1)} \right\} \\
V_j(e_2; x) &= (1 + s_j) \left\{ \frac{j(j-1)x^2}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2} \right\}
\end{aligned}$$

where $e_i(t) = t^i$. So for any $\lambda > 0$, we can see, that

$$\lambda |V_j(e_0; x) - e_0(x)| = \lambda |1 + s_j - 1| = \lambda s_j,$$

which implies

$$\rho^\varphi[\lambda(V_j(e_0) - e_0)] = \rho^\varphi(\lambda s_j) = \int_0^1 \varphi(\lambda s_j) dx = \varphi(\lambda s_j) = s_j \varphi(\lambda)$$

because of the definition of $\{s_j\}$. Since $\{s_j\}$ is convergent to 0 in the sense of power series method, for every $\lambda > 0$

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^\varphi(\lambda V_j(e_0) - e_0) = \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j s_j \varphi(\lambda) = 0.$$

Also

$$\begin{aligned}
\lambda |V_j(e_1; x) - e_1(x)| &= \lambda \left| x \left(\frac{j}{j+1} + \frac{j s_j}{j+1} - 1 \right) + \frac{1}{2(j+1)} + \frac{s_j}{2(j+1)} \right| \\
&\leq \lambda \left\{ \frac{3}{2(j+1)} + s_j \left(\frac{2j+1}{2(j+1)} \right) \right\},
\end{aligned}$$

we may write that

$$\begin{aligned}
\rho^\varphi[\lambda(V_j(e_1) - e_1)] &\leq \rho^\varphi \left(\lambda \left\{ s_j \left(\frac{2j+1}{2(j+1)} \right) + \frac{3}{2(j+1)} \right\} \right) \\
&\leq s_j \rho^\varphi \left(\lambda \left(\frac{2j+1}{j+1} \right) \right) + \rho^\varphi \left(\frac{3}{j+1} \right)
\end{aligned}$$

by the definitions of $\{s_j\}$ and ρ^φ . Since $\{\frac{2j+1}{j+1}\}$ is convergent, there exists a constant $M > 0$ such that $\{\frac{2j+1}{j+1}\} \leq M\}$, for every $j \in \mathbb{N}$. Then using the monotonicity of ρ^φ , we have

$$\rho^\varphi \left[\lambda \frac{2j+1}{j+1} \right] \leq \rho^\varphi(\lambda M)$$

for any $\lambda > 0$, which implies

$$\rho^\varphi[\lambda(V_j(e_1) - e_1)] \leq s_j \rho^\varphi(\lambda M) + \rho^\varphi \left(\frac{3\lambda}{j+1} \right) = s_j \varphi(\lambda M) + \varphi \left(\frac{3\lambda}{j+1} \right).$$

Since φ is continuous, we have $\lim_j \varphi(\frac{3\lambda}{j+1}) = \varphi(\lim_j \frac{3\lambda}{j+1}) = \varphi(0) = 0$. So we get $\varphi(\frac{3\lambda}{j+1})$ is convergent to 0 in the sense of power series method. Using this and by the definition of $\{s_j\}$, we obtain

$$\begin{aligned} & \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^\varphi(\lambda V_j(e_1) - e_1) \\ & \leq \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j [s_j \varphi(\lambda M) + \varphi(\frac{3\lambda}{j+1})] \\ & = \varphi(\lambda M) \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j s_j + \limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varphi(\frac{3\lambda}{j+1}) \\ & = 0 \end{aligned}$$

Finally, since

$$\begin{aligned} & \lambda |V_j(e_2; x) - e_2(x)| \\ & = \lambda \left| x^2 \frac{j(j-1)}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2} + s_j \frac{j(j-1)x^2}{(j+1)^2} + s_j \frac{2jx}{(j+1)^2} \right. \\ & \quad \left. + s_j \frac{1}{3(j+1)^2} - x^2 \right| \\ & \leq \lambda \left\{ \frac{15j+4}{3(j+1)^2} + s_j \left(\frac{3j^2+3j+1}{3(j+1)^2} \right) \right\}. \end{aligned}$$

Since $\{\frac{3j^2+3j+1}{3(j+1)^2}\}$ is convergent, there exists a constant $K > 0$ such that $|\frac{3j^2+3j+1}{3(j+1)^2}| \leq K$, for every $j \in \mathbb{N}$. Then using the monotonicity of ρ^φ and the definition of $\{s_j\}$, we have

$$\begin{aligned} \rho^\varphi[\lambda(V_j(e_2) - e_2)] & \leq \rho^\varphi \left(2\lambda \left(\frac{15j+4}{3(j+1)^2} \right) \right) + \rho^\varphi \left(2\lambda s_j \left(\frac{3j^2+3j+1}{3(j+1)^2} \right) \right) \\ & \leq \rho^\varphi \left(\lambda \left(\frac{30j+8}{3(j+1)^2} \right) \right) + \rho^\varphi(2\lambda s_j K), \end{aligned}$$

where which yields

$$\rho^\varphi[\lambda(V_j(e_2) - e_2)] \leq \varphi \left(\lambda \left(\frac{30j+8}{3(j+1)^2} \right) \right) + s_j \varphi(2\lambda K).$$

Since φ is continuous, we have $\lim_j \varphi \left(\lambda \left(\frac{30j+8}{3(j+1)^2} \right) \right) = \varphi \left(\lambda \lim_j \frac{30j+8}{3(j+1)^2} \right) = \varphi(0) = 0$. So we get $\varphi \left(\lambda \left(\frac{30j+8}{3(j+1)^2} \right) \right)$ is convergent to 0 in the sense of power series method. Using this and by the definition of $\{s_j\}$, we obtain

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^\varphi(\lambda V_j(e_2) - e_2) = 0, \text{ for every } \lambda > 0.$$

So we can say that our sequence $V := \{V_j\}$ satisfies all assumptions of Theorem 1. Therefore we conclude that

$$\limsup_{t \rightarrow R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^\varphi(\lambda_0 V_j(f) - f) = 0, \text{ for some } \lambda_0 > 0$$

holds for every $f \in L_\varphi^p(G)$ such that $f - g \in X_V$ for every $g \in C^\infty(G)$. However since $\{s_j\}$ is not convergent to zero, it is clear that $\{V_j(f)\}$ is not modularly convergent to f .

Note that

- in the case of $R = 1$, $p(t) = \frac{1}{1-t}$ and for $j \geq 0$, $p_j = 1$ the power series method coincides with Abel method which is a sequence-to-function transformation,
- in the case of $R = \infty$, $p(t) = e^t$ and for $j \geq 0$, $p_j = \frac{1}{j!}$ the power series method coincides with Borel method.

We can therefore give all of the theorems of this paper for Abel and Borel convergences.

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Current address: Department of Mathematics, Hitit University, Çorum, Turkey

E-mail address: tugbayurdakadim@hotmail.com