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A note on the generalized Matsumoto relation

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Abstract: We give an elementary proof of a relation, first discovered in its full generality by Korkmaz, in the mapping class group of a closed orientable surface. Our proof uses only the well-known relations between Dehn twists.

Key words: Mapping class groups, braid relation, chain relation

1. Introduction

Our aim here is to give an alternative proof of Theorem 3.4 of [3], given below. This theorem is a generalization of the Matsumoto relation in the mapping class group of a closed orientable surface of genus 2 obtained in [4], to the higher genus case. We will refer to this relation as the generalized Matsumoto relation. It gives a relation involving $2g+4$ (respectively $2g+10$) Dehn twists when the genus of the surface is even (respectively odd).

Throughout the paper we denote the isotopy class of the right-handed Dehn twist about a simple closed curve c by the same letter c. We use functional notation, that is, for any two mapping classes f and g, the multiplication fg means that g is applied first. Let Σ_g denote a closed connected orientable surface of genus g.

Theorem(Korkmaz). *In the mapping class group of Σ_g, the following relations between right Dehn twists hold (see Figures 1 and 2):*

(i) $(B_0 B_1 B_2 \cdots B_g \sigma)^2 = 1$ *if g is even,*

(ii) $(B_0 B_1 B_2 \cdots B_g a^2 b^2)^2 = 1$ *if g is odd.*

The above theorem is used to show that there are infinitely many pairwise nonhomeomorphic 4-manifolds that admit genus-g Lefschetz fibrations over S^2 but do not carry any complex structure with either orientation (see [3, 5]).

Recall that the hyperelliptic mapping class group of Σ_g is a quotient of the braid group B_{2g+2} on $2g+2$ strings. The quotient of the hyperelliptic mapping class group with the cyclic subgroup of order 2 generated by the hyperelliptic involution is isomorphic to the mapping class group of a sphere with $2g+2$ punctures. The hyperelliptic mapping class group is equal to the mapping class group when $g=2$. Using these facts, to

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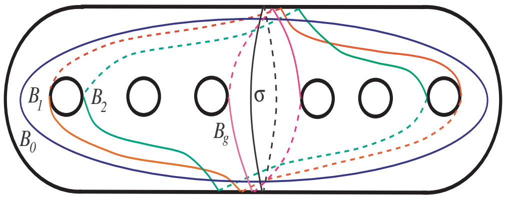

Figure 1. The curves B_i, when g is even.

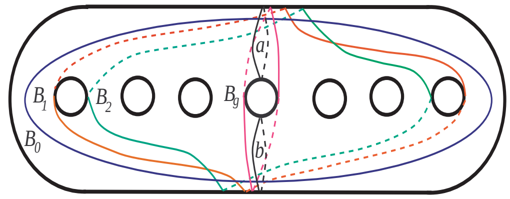

Figure 2. The curves B_i, when g is odd.

obtain the above-mentioned relations in the mapping class group, Korkmaz lifts Matsumoto's relation to the braid group B_6 and generalizes it to a relation in the braid group B_{2g+2}. He then projects it to the surface Σ_g to get these relations in the mapping class group of Σ_g.

In our main theorem, we obtain different set of curves. We then find a self-homeomorphism R of Σ_g, which takes B_i's to A_{g-i}'s, i.e. $R(B_i) = A_{g-i}$ for $0 \leq i \leq g$. Here is our main theorem:

Main Theorem. *In the mapping class group of* Σ_g*, the following relations hold:*

(i) $(A_gA_{g-1}\cdots A_0\sigma)^2 = 1$ *if* g *is even,*

(ii) $(A_gA_{g-1}\cdots A_0a^2b^2)^2 = 1$ *if* g *is odd.*

In Figures 3 and 4, the curves $A_0, A_1, \ldots, A_g$ are given for $g = 6$ and $g = 7$, respectively.

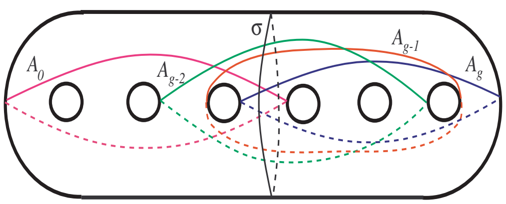

Figure 3. Curves A_i when g is even.

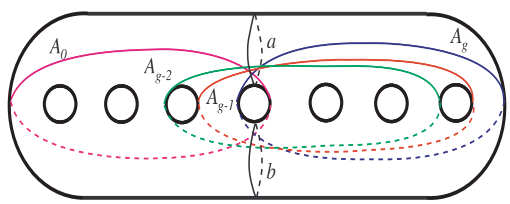

Figure 4. Curves A_i when g is odd.

In the proof, we only use the following well-known relations among Dehn twists. For completeness of the text we recall them here.

Commutativity Relation: If the geometric intersection number of the curves a and b is zero, then the Dehn twists about these curves commute, i.e. $ab = ba$.

Braid Relation: If the geometric intersection number of the curves a and b is 1, then we have $aba = bab$.

Chain Relation: If a and b (σ) are the boundary curves of a regular neighborhood of the chain of simple closed curves $c_1, c_2, \ldots c_k$ for k odd (for k even), then (see Figures 5 and 6)

(i) when k is odd $(c_kc_{k-1}\cdots c_2c_1)^{k+1} = ab$,

(ii) when k is even $(c_kc_{k-1}\cdots c_2c_1)^{2k+2} = \sigma$.

To make the text easier to follow, we underline the curves before and after we apply the above relations. We refer the reader to [1] for more details on the basic concepts of mapping class groups.

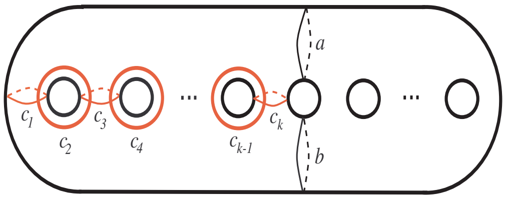

Figure 5. Chain relation for k odd.

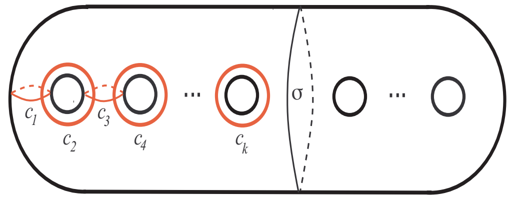

Figure 6. Chain relation for k even.

2. Proofs

In the following proof we generalize the techniques used in the proof of [2, Lemma 2.3] to arbitrary genera. Throughout this section let c_i denote the right-handed Dehn twist about the simple closed curve in Figure 7 for $i = 1, 2, \dots, 2g+1$.

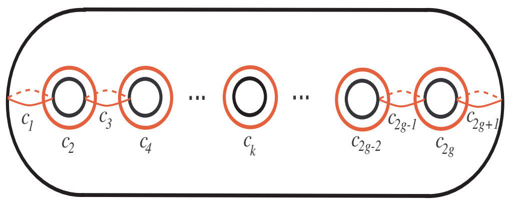

Figure 7. Genus g surface, Σ_g.

Lemma 2.1. *The product* $(c_{2g+1}c_{2g}c_{2g-1}\cdots c_2c_1)^n$ *can be expressed as*

$$\left(\prod_{i=-n+2}^{1} c_{2g+i} \prod_{i=-n+2}^{1} c_{2g+i-1} \cdots \prod_{i=-n+2}^{1} c_{g+i+1} \prod_{i=-n+2}^{1} c_{g+i}\right)(c_g c_{g-1}\cdots c_2 c_1)^n,$$

for $1 \le n \le g+1$.

Proof We proceed by induction on n. For $n = 1$, the statement is clear, that is we have

$$(c_{2g+1}c_{2g}c_{2g-1}\cdots c_2c_1) = (c_{2g+1}c_{2g}\cdots c_{g+1})(c_g c_{g-1}\cdots c_2c_1).$$

For $n = 2$, note first that the set of curves $\{c_{g-1}, c_{g-2}, \dots, c_2, c_1\}$ are disjoint from the set of curves $\{c_{2g+1}, c_{2g}, \dots, c_{g+2}, c_{g+1}\}$; hence Dehn twists about these curves commute. We have

$$\begin{aligned}
&(c_{2g+1}c_{2g}c_{2g-1}\cdots c_2c_1)^2\\
=&(c_{2g+1}c_{2g}\cdots c_g\underline{c_{g-1}\cdots c_2c_1})(c_{2g+1}c_{2g}\cdots c_{g+1}c_g c_{g-1}\cdots c_2c_1)\\
=&(c_{2g+1}c_{2g}c_{2g-1}\cdots c_{g+1}c_g)(c_{2g+1}c_{2g}\cdots c_{g+1}\underline{c_{g-1}\cdots c_2c_1}c_g c_{g-1}\cdots c_2c_1).
\end{aligned}$$

Applying the commutativity relation again to $c_g, c_{g+1}, \dots c_{2g-1}$ with this order,

$$\begin{aligned}
&(c_{2g+1}c_{2g}\underline{c_{2g-1}}\cdots \underline{c_{g+1}}\ \underline{c_g})(c_{2g+1}c_{2g}\cdots c_{g+1}c_{g-1}\cdots c_2c_1c_gc_{g-1}\cdots c_2c_1)\\
=&(c_{2g+1}c_{2g})(c_{2g+1}\underline{c_{2g-1}}c_{2g}\cdots \underline{c_{g+1}}c_{g+2}\underline{c_g}c_{g+1}c_{g-1}\cdots c_2c_1c_gc_{g-1}\cdots c_2c_1).
\end{aligned}$$

We can regroup these terms in the following way:

$$(c_{2g+1}c_{2g}c_{2g+1}c_{2g-1}c_{2g}\cdots c_{g+1}c_{g+2}c_gc_{g+1}c_{g-1}\cdots c_2c_1)(c_gc_{g-1}\cdots c_2c_1).$$

Applying the braid relation successively to the underlined terms

$$
\begin{aligned}
&= \left(\underline{c_{2g}c_{2g+1}c_{2g}}c_{2g-1}c_{2g}c_{2g-2}\cdots c_{g+2}c_{g}c_{g+1}c_{g-1}\cdots c_{2}c_{1}\right)(c_{g}c_{g-1}\cdots c_{2}c_{1})\\
&= \left(c_{2g}c_{2g+1}\underline{c_{2g-1}c_{2g}c_{2g-1}}c_{2g-2}\cdots c_{g+2}c_{g}c_{g+1}c_{g-1}\cdots c_{2}c_{1}\right)(c_{g}c_{g-1}\cdots c_{2}c_{1})\\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&= \left(c_{2g}c_{2g+1}c_{2g-1}c_{2g}\cdots \underline{c_{g+1}c_{g+2}c_{g+1}}c_{g}c_{g+1}c_{g-1}\cdots c_{2}c_{1}\right)(c_{g}c_{g-1}\cdots c_{2}c_{1})\\
&= \left(c_{2g}c_{2g+1}c_{2g-1}c_{2g}c_{2g-2}c_{2g-1}\cdots \underline{c_{g}c_{g+1}c_{g}}c_{g-1}\cdots c_{2}c_{1}\right)(c_{g}c_{g-1}\cdots c_{2}c_{1}).
\end{aligned}
$$

The terms on the last line can be rewritten in the following way:

$$
\begin{aligned}
&(c_{2g}c_{2g+1}c_{2g-1}c_{2g}c_{2g-2}c_{2g-1}\cdots c_{g}c_{g+1})(c_{g}c_{g-1}\cdots c_{2}c_{1})(c_{g}c_{g-1}\cdots c_{2}c_{1})\\
&= ((c_{2g}c_{2g+1})(c_{2g-1}c_{2g})(c_{2g-2}c_{2g-1})\cdots(c_{g}c_{g+1}))(c_{g}c_{g-1}\cdots c_{2}c_{1})^{2}\\
&= \left(\prod_{i=0}^{1} c_{2g+i}\prod_{i=0}^{1} c_{2g+i-1}\prod_{i=0}^{1} c_{2g+i-2}\cdots\prod_{i=0}^{1} c_{2g+i-g}\right)(c_{g}c_{g-1}\cdots c_{2}c_{1})^{2}.
\end{aligned}
$$

Thus we are done for $n=2$.

Now assuming that the statement is true for k, we show that it is also true for $k+1$, i.e. we want to show

$$
(c_{2g+1}c_{2g}\cdots c_{1})^{k+1} = \left(\prod_{i=-k+1}^{1} c_{2g+i}\cdots \prod_{i=-k+1}^{1} c_{g+i+1}\prod_{i=-k+1}^{1} c_{g+i}\right)(c_{g}c_{g-1}\cdots c_{1})^{k+1}. \tag{1}
$$

Before we proceed, to simplify the notation and make the proofs easier to follow let us introduce P_k^i for the index decreasing product

$$
P_k^i := c_k c_{k-1}\cdots c_{i+1}c_i \quad \text{and} \quad P_{\overline{k}}^{\overline{i}} := \overline{c}_k\overline{c}_{k-1}\cdots\overline{c}_{i+1}\overline{c}_i \quad \text{for } i\le k,
$$

and Q_k^l for the index increasing product

$$
Q_k^l := c_k c_{k+1}\cdots c_{l-1}c_l \quad \text{and} \quad Q_{\overline{k}}^{\overline{l}} := \overline{c}_k\overline{c}_{k+1}\cdots\overline{c}_{l-1}\overline{c}_l \quad \text{for } k\le l,
$$

where $\overline{c}_k$ denotes the left-handed Dehn twist about the curve c_k. Using these notations, we can rephrase the equation (1) that we want to prove as

$$
\left(P_{2g+1}^{1}\right)^{k+1} = \left(Q_{2g-k+1}^{2g+1}\cdots Q_{g-k+2}^{g+2}Q_{g-k+1}^{g+1}\right)\left(P_{g}^{1}\right)^{k+1}.
$$

By the induction assumption on k, we have

$$
\left(P_{2g+1}^{1}\right)^{k} = \left(Q_{2g-k+2}^{2g+1}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}\right)\left(P_{g}^{1}\right)^{k}.
$$

Since we have $\left(P_{2g+1}^{1}\right)^{k+1}=P_{2g+1}^{1}\left(P_{2g+1}^{1}\right)^{k}$, we get

$$\begin{aligned}\left(P_{2g+1}^{1}\right)^{k+1} &= \left(P_{2g+1}^{1}\right)\left(Q_{2g-k+2}^{2g+1}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}\right)\left(P_{g}^{1}\right)^{k}\\ &= \left(P_{2g+1}^{2g-k+2}P_{2g-k+1}^{g-k+1}P_{g-k}^{1}\right)\left(Q_{2g-k+2}^{2g+1}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}\right)\left(P_{g}^{1}\right)^{k}.\end{aligned}$$

Since the curves in the product $P_{g-k}^{1}=c_{g-k}\cdots c_2c_1$ are disjoint from all the curves $c_{g-k+2},c_{g-k+3},\ldots,c_{2g}$, c_{2g+1} in the product $Q_{2g-k+2}^{2g+1}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}$, by commutativity of the Dehn twists about these curves,

$$\begin{aligned}&\left(P_{2g+1}^{2g-k+2}P_{2g-k+1}^{g-k+1}\underline{P_{g-k}^{1}}\right)\left(Q_{2g-k+2}^{2g+1}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}\right)\left(P_{g}^{1}\right)^{k}\\ &= \left(P_{2g+1}^{2g-k+2}P_{2g-k+1}^{g-k+1}\right)\left(Q_{2g-k+2}^{2g+1}Q_{2g-k+1}^{2g}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}\right)\underline{P_{g-k}^{1}}\left(P_{g}^{1}\right)^{k}.\end{aligned}$$

Similarly again by commutativity, we can write the Dehn twist about the product of the curves $P_{2g-k+1}^{g-k+1}=c_{2g-k+1}\cdots c_{g-k+2}c_{g-k+1}$ as follows:

$$\begin{aligned}&\left(P_{2g+1}^{2g-k+2}\underline{P_{2g-k+1}^{g-k+1}}\right)\left(Q_{2g-k+2}^{2g+1}Q_{2g-k+1}^{2g}\cdots Q_{g-k+3}^{g+2}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k}\\ &= \left(P_{2g+1}^{2g-k+2}\right)\left(\underline{c_{2g-k+1}}Q_{2g-k+2}^{2g+1}\underline{c_{2g-k}}Q_{2g-k+1}^{2g}\cdots \underline{c_{g-k+2}}Q_{g-k+3}^{g+2}\underline{c_{g-k+1}}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k}.\end{aligned}$$

Applying the braid relation, $c_{2g-k+2}c_{2g-k+1}c_{2g-k+2}=c_{2g-k+1}c_{2g-k+2}c_{2g-k+1}$,

$$\begin{aligned}&\left(P_{2g+1}^{2g-k+3}\right)\underline{c_{2g-k+2}}\left(\underline{c_{2g-k+1}c_{2g-k+2}}Q_{2g-k+3}^{2g+1}\cdots c_{g-k+2}Q_{g-k+3}^{g+2}c_{g-k+1}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k}\\ &= \left(P_{2g+1}^{2g-k+3}\right)\left(\underline{c_{2g-k+1}c_{2g-k+2}c_{2g-k+1}}Q_{2g-k+3}^{2g+1}\cdots c_{g-k+2}Q_{g-k+3}^{g+2}c_{g-k+1}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k}.\end{aligned}$$

Since the curve c_{2g-k+1} is disjoint from all the curves in Q_{2g-k+3}^{2g+1}, by commutativity we have

$$\begin{aligned}&\left(P_{2g+1}^{2g-k+3}\right)\left(c_{2g-k+1}c_{2g-k+2}\underline{c_{2g-k+1}}Q_{2g-k+3}^{2g+1}\cdots c_{g-k+2}Q_{g-k+3}^{g+2}c_{g-k+1}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k}\\ &= \left(P_{2g+1}^{2g-k+3}\right)\left(c_{2g-k+1}c_{2g-k+2}Q_{2g-k+3}^{2g+1}\underline{c_{2g-k+1}}\cdots c_{g-k+2}Q_{g-k+3}^{g+2}c_{g-k+1}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k},\end{aligned}$$

which can also be written as

$$\left(P_{2g+1}^{2g-k+3}\right)\left(Q_{2g-k+1}^{2g+1}c_{2g-k+1}c_{2g-k}Q_{2g-k+1}^{2g}\cdots c_{g-k+2}Q_{g-k+3}^{g+2}c_{g-k+1}Q_{g-k+2}^{g+1}\right)P_{g-k}^{1}\left(P_{g}^{1}\right)^{k}.$$

Applying braid relations

$$\begin{aligned}c_{2g-k+1}c_{2g-k}c_{2g-k+1} &= c_{2g-k}c_{2g-k+1}c_{2g-k}\\ \vdots\qquad & \qquad\vdots\\ c_{g-k+3}c_{g-k+2}c_{g-k+3} &= c_{g-k+2}c_{g-k+3}c_{g-k+2}\end{aligned}$$

and commutativity succesively, and by using our increasing product notation we get

$$\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\underline{c_{2g-k+1}}c_{2g-k}Q^{2g}_{2g-k+1}\cdots c_{g-k+2}Q^{g+2}_{g-k+3}c_{g-k+1}Q^{g+1}_{g-k+2}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k}$$

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}c_{2g-k}Q^{2g}_{2g-k+1}\cdots c_{g-k+2}Q^{g+2}_{g-k+3}\underline{c_{g-k+2}}c_{g-k+1}Q^{g+1}_{g-k+2}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k},$$

which can also be written as

$$\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}Q^{2g}_{2g-k}\cdots Q^{g+2}_{g-k+2}\underline{c_{g-k+2}}c_{g-k+1}Q^{g+1}_{g-k+2}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k}.$$

Let us write the final braid and commutativity relation explicitly. Applying the braid relation $c_{g-k+2}c_{g-k+1}c_{g-k+2} = c_{g-k+1}c_{g-k+2}c_{g-k+1}$, we obtain

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}\underline{c_{g-k+2}c_{g-k+1}c_{g-k+2}}Q^{g+1}_{g-k+3}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k}$$

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}\underline{c_{g-k+1}c_{g-k+2}c_{g-k+1}}Q^{g+1}_{g-k+3}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k},$$

and now applying the commutativity relation:

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}c_{g-k+1}c_{g-k+2}\underline{c_{g-k+1}}Q^{g+1}_{g-k+3}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k}$$

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}c_{g-k+1}c_{g-k+2}Q^{g+1}_{g-k+3}\underline{c_{g-k+1}}\right)P^{1}_{g-k}\left(P^{1}_{g}\right)^{k}.$$

After this procedure the Dehn twist c_{2g-k+2} becomes c_{g-k+1} and hence we obtain P^{1}_{g-k+1} on the immediate left-hand side of $\left(P^{1}_{g}\right)^{k}$.

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}c_{g-k+1}c_{g-k+2}Q^{g+1}_{g-k+3}\right)P^{1}_{g-k+1}\left(P^{1}_{g}\right)^{k}.$$

Therefore, by using our increasing index product notation, we can write the above expression as

$$=\left(P^{2g-k+3}_{2g+1}\right)\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}Q^{g+1}_{g-k+1}\right)P^{1}_{g-k+1}\left(P^{1}_{g}\right)^{k}.$$

Applying the same procedure to the Dehn twists $c_{2g-k+3},\ldots,c_{2g},c_{2g+1}$ in $\left(P^{2g-k+3}_{2g+1}\right)$, they become c_{g-k+2}, $\ldots,c_{g-1},c_{g}$ respectively and as a result we obtain P^{1}_{g}.

$$=\left(Q^{2g+1}_{2g-k+1}\cdots Q^{g+2}_{g-k+2}Q^{g+1}_{g-k+1}\right)c_{g}c_{g-1}\cdots c_{g-k+2}P^{1}_{g-k+1}\left(P^{1}_{g}\right)^{k}.$$

Therefore we have the desired result

$$\left(P^{1}_{2g+1}\right)^{k+1}=\left(Q^{2g+1}_{2g-k+1}Q^{2g}_{2g-k}\cdots Q^{g+2}_{g-k+2}Q^{g+1}_{g-k+1}\right)\left(P^{1}_{g}\right)^{k+1}.$$

Lemma 2.2. *In the mapping class group of Σ_g, we have*

$$(c_1c_2\cdots c_g)^{g+1} = (c_gc_{g-1}\cdots c_2c_1)^{g+1}$$

Proof When g is odd, by the chain relation (see Figure 5) we have

$$(c_1c_2\cdots c_g)^{g+1} = ab = (c_gc_{g-1}\cdots c_1)^{g+1}.$$

Thus we are done when g is odd.

When g is even, i.e. $g = 2k$, by using the commutativity and braid relations

$$\begin{aligned}
(c_1c_2\cdots c_{2k})^{2k+1} &= \overbrace{(c_1c_2\cdots \underline{c_{2k}})(c_1c_2\cdots c_{2k})\cdots(c_1c_2\cdots c_{2k})}^{2k-\text{many}}(c_1c_2\cdots c_{2k})\\
&= (c_1c_2\cdots c_{2k-1})\overbrace{(c_1c_2\cdots \underline{c_{2k}})\cdots(c_1c_2\cdots c_{2k})}^{(2k-1)-\text{many}}\underline{c_1}(c_1c_2\cdots c_{2k})\\
&= (c_1c_2\cdots c_{2k-1})(c_1c_2\cdots c_{2k-1})\cdots(c_1c_2\cdots c_{2k})\underline{c_2}c_1(c_1c_2\cdots c_{2k})\\
&\ \ \vdots\\
&= \overbrace{(c_1c_2\cdots c_{2k-1})\cdots(c_1c_2\cdots c_{2k-1})}^{2k-\text{many}}c_{2k}c_{2k-1}\cdots c_2c_1(c_1c_2\cdots c_{2k})\\
&= (c_1c_2\cdots c_{2k-1})^{2k}c_{2k}c_{2k-1}\cdots c_2c_1c_1c_2\cdots c_{2k-1}c_{2k}.
\end{aligned}$$

By the previous case, the above product is equal to

$$= (c_{2k-1}\cdots c_2c_1)^{2k}c_{2k}c_{2k-1}\cdots c_2c_1c_1c_2\cdots c_{2k-1}c_{2k}.$$

Rephrasing, we have

$$= \overbrace{(P^1_{2k-1}P^1_{2k-1}\cdots P^1_{2k-1})}^{2k-\text{many}}c_{2k}c_{2k-1}\cdots c_2c_1Q^{2k}_1$$

By commutativity we can move c_1 in P^1_{2k-1} and write the right-hand side as in the following:

$$\begin{aligned}
&= (P^1_{2k-1}P^1_{2k-1}\cdots P^2_{2k-1}\underline{c_1})c_{2k}c_{2k-1}\cdots c_2c_1Q^{2k}_1\\
&= (P^1_{2k-1}P^1_{2k-1}\cdots P^2_{2k-1})c_{2k}c_{2k-1}\cdots \underline{c_1}c_2c_1Q^{2k}_1.
\end{aligned}$$

By applying braid relation $c_1c_2c_1 = c_2c_1c_2$ we can write the right-hand side as follows:

$$\begin{aligned}
&= (P^1_{2k-1}P^1_{2k-1}\cdots P^2_{2k-1})c_{2k}c_{2k-1}\cdots \underline{c_1c_2c_1}Q^{2k}_1\\
&= (P^1_{2k-1}P^1_{2k-1}\cdots P^2_{2k-1})c_{2k}c_{2k-1}\cdots \underline{c_2c_1c_2}Q^{2k}_1.
\end{aligned}$$

Similarly we can move the curve c_2 in the product P^2_{2k-1} as in the following:

$$\begin{aligned}
&= (P^1_{2k-1}P^1_{2k-1}\cdots P^3_{2k-1}\underline{c_2})c_{2k}c_{2k-1}\cdots c_2c_1c_2Q^{2k}_1\\
&= (P^1_{2k-1}P^1_{2k-1}\cdots P^3_{2k-1})c_{2k}c_{2k-1}\cdots \underline{c_2}c_3c_2c_1c_2Q^{2k}_1,
\end{aligned}$$

and after braid relation $c_2c_3c_2 = c_3c_2c_3$ we get

$$= \left(P^1_{2k-1}P^1_{2k-1}\cdots P^3_{2k-1}\right) c_{2k}c_{2k-1}\cdots \underline{c_2c_3c_2}c_1c_2Q^{2k}_1$$

$$= \left(P^1_{2k-1}P^1_{2k-1}\cdots P^3_{2k-1}\right) c_{2k}c_{2k-1}\cdots \underline{c_3c_2c_3}c_1c_2Q^{2k}_1.$$

Since the Dehn twists c_3 and c_1 commute, we have

$$= \left(P^1_{2k-1}P^1_{2k-1}\cdots P^3_{2k-1}\right) c_{2k}c_{2k-1}\cdots c_3c_2\underline{c_3}c_1c_2Q^{2k}_1.$$

$$= \left(P^1_{2k-1}P^1_{2k-1}\cdots P^3_{2k-1}\right) c_{2k}c_{2k-1}\cdots c_3c_2c_1\underline{c_3}c_2Q^{2k}_1.$$

Moreover, applying commutativity and braid relations to the Dehn twists in the product P^3_{2k-1} repeatedly we get

$$= \overbrace{\left(P^1_{2k-1}P^1_{2k-1}\cdots P^1_{2k-1}\right)}^{(2k-1)-\text{many}} P^1_{2k}P^2_{2k}Q^{2k}_1.$$

Similarly when we apply the same operations to the $2k-1$ Dehn twists in the remaining $2k-1$ products P^1_{2k-1}, we obtain

$$= P^1_{2k}\left(P^2_{2k}\right)^{2k} Q^{2k}_1.$$

Now, applying commutativity and braid relations,

$$\begin{aligned}
(c_{2k}c_{2k-1}\cdots c_1)^{2k} &= \overbrace{\left(c_{2k}c_{2k-1}\cdots \underline{c_1}\right)\left(c_{2k}c_{2k-1}\cdots c_1\right)\cdots\left(c_{2k}c_{2k-1}\cdots c_1\right)\left(c_{2k}c_{2k-1}\cdots c_1\right)}^{2k-\text{many}} \\
&= \left(c_{2k}c_{2k-1}\cdots c_2\right)\overbrace{\left(c_{2k}c_{2k-1}\cdots \underline{c_1}\right)\cdots\left(c_{2k}c_{2k-1}\cdots c_1\right)\left(c_{2k}c_{2k-1}\cdots c_1\right)}^{(2k-1)-\text{many}}\underline{c_{2k}} \\
&= \left(c_{2k}c_{2k-1}\cdots c_2\right)\left(c_{2k}c_{2k-1}\cdots c_2\right)\cdots\left(c_{2k}c_{2k-1}\cdots c_1\right)\left(c_{2k}c_{2k-1}\cdots c_1\right)\underline{c_{2k-1}}c_{2k} \\
&\;\vdots \\
&= \overbrace{\left(c_{2k}c_{2k-1}\cdots c_2\right)\cdots\left(c_{2k}c_{2k-1}\cdots c_2\right)\left(c_{2k}c_{2k-1}\cdots c_2\right)}^{2k-\text{many}}\underline{c_1}c_2\cdots c_{2k} \\
&= \left(c_{2k}c_{2k-1}\cdots c_2\right)^{2k} c_1c_2\cdots c_{2k}.
\end{aligned}$$

Therefore we get $\left(P^2_{2k}\right)^{2k} Q^{2k}_1 = \left(P^1_{2k}\right)^{2k}$.

Let us summarize quickly what we have done:

$$\begin{aligned}
&\left(c_1c_2\cdots c_{2k}\right)^{2k+1} \\
&= \left(c_1c_2\cdots c_{2k-1}\right)^{2k} c_{2k}c_{2k-1}\cdots c_2c_1c_1c_2\cdots c_{2k-1}c_{2k} \\
&= P^1_{2k}\left(P^2_{2k}\right)^{2k} Q^{2k}_1 \\
&= P^1_{2k}\left(P^1_{2k}\right)^{2k} \\
&= \left(P^1_{2k}\right)^{2k+1} = \left(c_{2k}\cdots c_2c_1\right)^{2k+1}.
\end{aligned}$$

Lemma 2.3. *In the mapping class group of* Σ_g*, we have the following relations (see Figure 8*)*

(i) $(c_{2g+1}c_{2g}\cdots c_2c_1)^{g+1} = \left(Q_{g+1}^{2g+1}P_{2g}^{\overline{g+1}}Q_g^{2g}P_{\overline{2g-1}}^{\overline{g}}\cdots Q_1^{g+1}P_{\overline{g-1}}^{\overline{1}}\right)a^2b^2,$ *if* g *is odd.*

(ii) $(c_{2g+1}c_{2g}\cdots c_2c_1)^{g+1} = \left(Q_{g+1}^{2g+1}P_{2g}^{\overline{g+1}}Q_g^{2g}P_{\overline{2g-1}}^{\overline{g}}\cdots Q_1^{g+1}P_{\overline{g-1}}^{\overline{1}}\right)\sigma,$ *if* g *is even.*

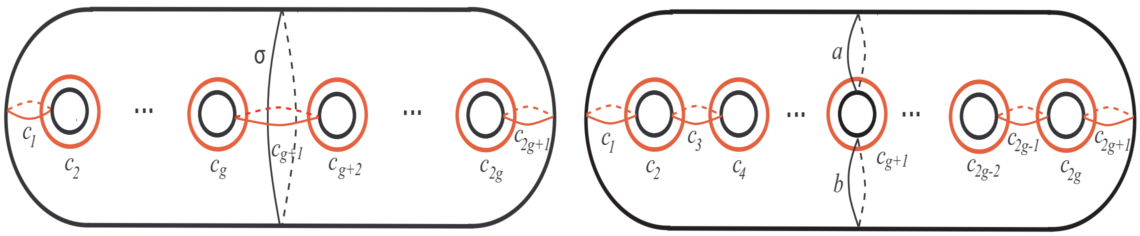

Figure 8.

Proof By Lemma 2.1, for $n = g+1$ we have

$$\begin{aligned}&(c_{2g+1}c_{2g}\cdots c_2c_1)^{g+1}\\ &= (c_{g+1}\cdots c_{2g}c_{2g+1})(c_g\cdots c_{2g-1}c_{2g})\cdots(c_1\cdots c_gc_{g+1})(c_g\cdots c_2c_1)^{g+1}.\end{aligned}$$

This can be written also as

$$\left(P_{2g+1}^1\right)^{g+1} = \left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g+1}\right)\left(P_g^1\right)^{g+1}.$$

Let us denote $\left(P_g^1\right)^{g+1}$ by I, and multiply the right-hand side of the relation by I and $\overline{I}$ as in the following:

$$\left(P_{2g+1}^1\right)^{g+1} = \left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g+1}\right)\overline{I}I\left(P_g^1\right)^{g+1}$$

By Lemma 2.2, we have $\overline{I} = (\overline{c}_g\overline{c}_{g-1}\cdots\overline{c}_1)^{g+1} = \left(P_{\overline{g}}^{\overline{1}}\right)^{g+1}$, and so we can write the right-hand side of the above equation as follows:

$$\begin{aligned}&\left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g+1}\right)\left(P_{\overline{g}}^{\overline{1}}\right)^{g+1}\left(P_g^1\right)^{g+1}\left(P_g^1\right)^{g+1}\\ &= \left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g+1}\right)\left(P_{\overline{g}}^{\overline{1}}\right)^{g+1}\left(P_g^1\right)^{2g+2}\end{aligned}$$

Let us rewrite the terms in the product Q_1^{g+1} as in the following:

$$= \left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g-1}c_gc_{g+1}\right)\left(\overline{c}_gP_{\overline{g-1}}^{\overline{1}}(P_{\overline{g}}^{\overline{1}})^g\right)\left(P_g^1\right)^{2g+2}$$

By applying braid relation $c_gc_{g+1}\overline{c}_g = \overline{c}_{g+1}c_gc_{g+1}$ we can write the right-hand side as follows:

$$\begin{aligned}&= \left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g-1}\right)\underline{c_gc_{g+1}\overline{c}_g}\left(P_{\overline{g-1}}^{\overline{1}}(P_{\overline{g}}^{\overline{1}})^g\right)\left(P_g^1\right)^{2g+2}\\ &= \left(Q_{g+1}^{2g+1}Q_g^{2g}\cdots Q_2^{g+2}Q_1^{g-1}\right)\underline{\overline{c}_{g+1}c_gc_{g+1}}\left(P_{\overline{g-1}}^{\overline{1}}(P_{\overline{g}}^{\overline{1}})^g\right)\left(P_g^1\right)^{2g+2}\end{aligned}$$

*We thank the referee for providing us with this figure.

By commutativity of Dehn twist, we can move the curve $\overline{c}_{g+1}$ to the left as follows:

$$= \left(Q_{g+1}^{2g+1} Q_{g}^{2g} \cdots Q_{2}^{g+2} Q_{1}^{g-1}\right) \underline{\overline{c}_{g+1}} c_g c_{g+1} \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2}$$

$$= \left(Q_{g+1}^{2g+1} Q_{g}^{2g} \cdots Q_{2}^{g+2} \underline{\overline{c}_{g+1}} Q_{1}^{g-1}\right) c_g c_{g+1} \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2},$$

which can also be written as

$$\left(Q_{g+1}^{2g+1} Q_{g}^{2g} \cdots Q_{2}^{g+2} \underline{\overline{c}_{g+1}} Q_{1}^{g+1}\right) \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2}.$$

We can write some of the terms in the product Q_2^{g+2} explicitly as in the following:

$$= \left(Q_{g+1}^{2g+1} Q_{g}^{2g} \cdots Q_{2}^{g} c_{g+1} c_{g+2} \overline{c}_{g+1} Q_{1}^{g+1}\right) \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2}$$

Now again we can apply braid relation $c_{g+1}c_{g+2}\overline{c}_{g+1} = \overline{c}_{g+2}c_{g+1}c_{g+2}$ and obtain the following:

$$= \left(Q_{g+1}^{2g+1} Q_{g}^{2g} \cdots Q_{2}^{g} \underline{c_{g+1} c_{g+2} \overline{c}_{g+1}} Q_{1}^{g+1}\right) \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2}$$

$$= \left(Q_{g+1}^{2g+1} Q_{g}^{2g} \cdots Q_{2}^{g} \underline{\overline{c}_{g+2} c_{g+1} c_{g+2}} Q_{1}^{g+1}\right) \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2}$$

By applying commutativity and braid relation $g-1$ more times we get

$$= \left(Q_{g+1}^{2g+1} \underline{\overline{c}_{2g}} Q_{g}^{2g} \cdots Q_{2}^{g+2} Q_{1}^{g+1}\right) \left(P_{\underline{g-1}}^{\overline{1}} (P_{\underline{g}}^{\overline{1}})^g\right) \left(P_g^1\right)^{2g+2}$$

We have started with $\overline{c}_g$, applied braid relation and commutativity repeatedly, and obtained $\overline{c}_{2g}$. Similarly we apply the same operations to the Dehn twists $\overline{c}_{g-1}, \overline{c}_{g-2}, \ldots, \overline{c}_2, \overline{c}_1$ in $P_{\underline{g-1}}^{\overline{1}}$ and obtain $\overline{c}_{2g-1}, \overline{c}_{2g-2}, \ldots, \overline{c}_{g+2}, \overline{c}_{g+1}$.

Then we can write the above equation as in the following:

$$\left(Q_{g+1}^{2g+1} \underline{\overline{c}_{2g}\overline{c}_{2g-1}} \cdots \underline{\overline{c}_{g+1}} Q_{g}^{2g} \cdots Q_{2}^{g+2} Q_{1}^{g+1}\right) \left(P_{\underline{g}}^{\overline{1}}\right)^g \left(P_g^1\right)^{2g+2}$$

$$= \left(Q_{g+1}^{2g+1} P_{\underline{2g}}^{\overline{g+1}} Q_{g}^{2g} \cdots Q_{2}^{g+2} Q_{1}^{g+1}\right) \left(P_{\underline{g}}^{\overline{1}}\right)^g \left(P_g^1\right)^{2g+2}$$

We can apply braid relation and commutativity in the same way to the Dehn twists in $\left(P_{\underline{g}}^{\overline{1}}\right)^g$ and obtain the following:

$$\left(Q_{g+1}^{2g+1} P_{\underline{2g}}^{\overline{g+1}} Q_{g}^{2g} P_{\underline{2g-1}}^{\overline{g}} \cdots Q_{2}^{g+2} P_{\underline{g+1}}^{\overline{2}} Q_{1}^{g+1} P_{\underline{g}}^{\overline{1}}\right) \left(P_g^1\right)^{2g+2}.$$

Using $k-$chain relation for $k=g$, we get the following:

When g is odd

$$\left(P_{2g+1}^{1}\right)^{g+1}$$

$$= \left(Q_{g+1}^{2g+1} P_{\underline{2g}}^{\overline{g+1}} Q_{g}^{2g} P_{\underline{2g-1}}^{\overline{g}} \cdots Q_{2}^{g+2} P_{\underline{g+1}}^{\overline{2}} Q_{1}^{g+1} P_{\underline{g}}^{\overline{1}}\right) \left(P_g^1\right)^{2g+2}$$

$$= \left(Q_{g+1}^{2g+1} P_{\underline{2g}}^{\overline{g+1}} Q_{g}^{2g} P_{\underline{2g-1}}^{\overline{g}} \cdots Q_{2}^{g+2} P_{\underline{g+1}}^{\overline{2}} Q_{1}^{g+1} P_{\underline{g}}^{\overline{1}}\right) a^2 b^2$$

When g is even

$$
\begin{aligned}
&\left(P^{1}_{2g+1}\right)^{g+1}\\
&=\left(Q^{2g+1}_{g+1}P^{\overline{g+1}}_{\overline{2g}}Q^{2g}_{g}P^{\overline{g}}_{\overline{2g-1}}\cdots Q^{g+2}_{2}P^{\overline{2}}_{\overline{g+1}}Q^{g+1}_{1}P^{\overline{1}}_{\overline{g}}\right)\left(P^{1}_{g}\right)^{2g+2}\\
&=\left(Q^{2g+1}_{g+1}P^{\overline{g+1}}_{\overline{2g}}Q^{2g}_{g}P^{\overline{g-1}}_{\overline{2g}}\cdots Q^{g+2}_{2}P^{\overline{2}}_{\overline{g+1}}Q^{g+1}_{1}P^{\overline{1}}_{\overline{g}}\right)\sigma
\end{aligned}
$$

Thus we are done.

Let us denote the product of Dehn twists of the previous lemma as in the following:

$$
\begin{aligned}
Q^{g+1}_{1}P^{\overline{1}}_{\overline{g}}&=(c_1c_2\cdots c_{g+1}\overline{c}_g\overline{c}_{g-1}\cdots\overline{c}_1):=\mathbf{A_0},\\
Q^{g+2}_{2}P^{\overline{2}}_{\overline{g+1}}&=(c_2c_3\cdots c_{g+2}\overline{c}_{g+1}\overline{c}_g\cdots\overline{c}_2):=\mathbf{A_1},\\
&\vdots\\
Q^{2g+1}_{g+1}P^{\overline{g+1}}_{\overline{2g}}&=(c_{g+1}c_{g+2}\cdots c_{2g+1}\overline{c}_{2g}\overline{c}_{2g-1}\cdots\overline{c}_{g+1}):=\mathbf{A_g}.
\end{aligned}
$$

Note that the mapping class group element $\mathbf{A_i}$ is the Dehn twist around the image of the curve c_{i+g+1} under the product $c_{i+1}c_{i+2}\cdots c_{i+g}$ (see Fact 3.7 in [1]), which is the curve A_i in Figures 3 and 4.

Main Theorem. *In the mapping class group of Σ_g, the following relations hold (see Figures 3 and 4):*

(i) $(A_gA_{g-1}\cdots A_0\sigma)^2=1$ *if* g *is even,*

(ii) $(A_gA_{g-1}\cdots A_0a^2b^2)^2=1$ *if* g *is odd.*

Proof By Lemma 2.3 for g is even we have

$$
\begin{aligned}
(c_{2g+1}c_{2g}\cdots c_2c_1)^{g+1}&=\left(Q^{2g+1}_{g+1}P^{\overline{g+1}}_{\overline{2g}}Q^{2g}_{g}P^{\overline{g}}_{\overline{2g-1}}\cdots Q^{g+1}_{1}P^{\overline{1}}_{\overline{g-1}}\right)\sigma\\
&=(A_gA_{g-1}\ldots A_0)\,\sigma
\end{aligned}
$$

and for g is odd we have

$$
\begin{aligned}
(c_{2g+1}c_{2g}\cdots c_2c_1)^{g+1}&=\left(Q^{2g+1}_{g+1}P^{\overline{g+1}}_{\overline{2g}}Q^{2g}_{g}P^{\overline{g}}_{\overline{2g-1}}\cdots Q^{g+1}_{1}P^{\overline{1}}_{\overline{g-1}}\right)a^2b^2\\
&=(A_gA_{g-1}\ldots A_0)\,a^2b^2
\end{aligned}
$$

Now, to finish the proof it is enough to take the squares of both sides of the above relations and use the chain relation.

To see that the above proof is actually an alternative proof of Theorem 3.4 of [3], first recall that in Figures 1 and 2, the curves in the Matsumoto relation were given. Then one can observe that the curves B_i in the Matsumoto relation and the curves A_{g-i} are related in the following way. Let

$$R=(\overline{c}_1\overline{c}_2\cdots\overline{c}_g)(\overline{c}_1\overline{c}_2\cdots\overline{c}_{g-1})(\overline{c}_1\overline{c}_2\cdots\overline{c}_{g-2})\cdots(\overline{c}_1\overline{c}_2)\overline{c}_1.$$

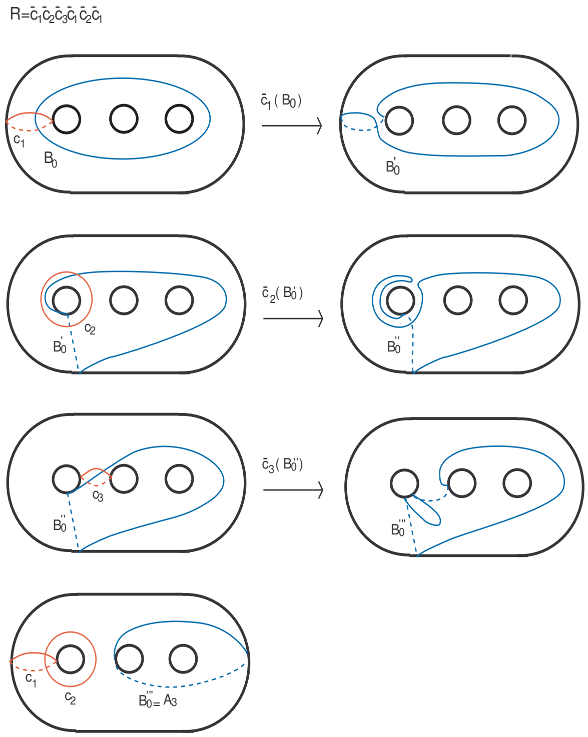

Figure 9. $R(B_0) = A_3$ in genus 3 surface.

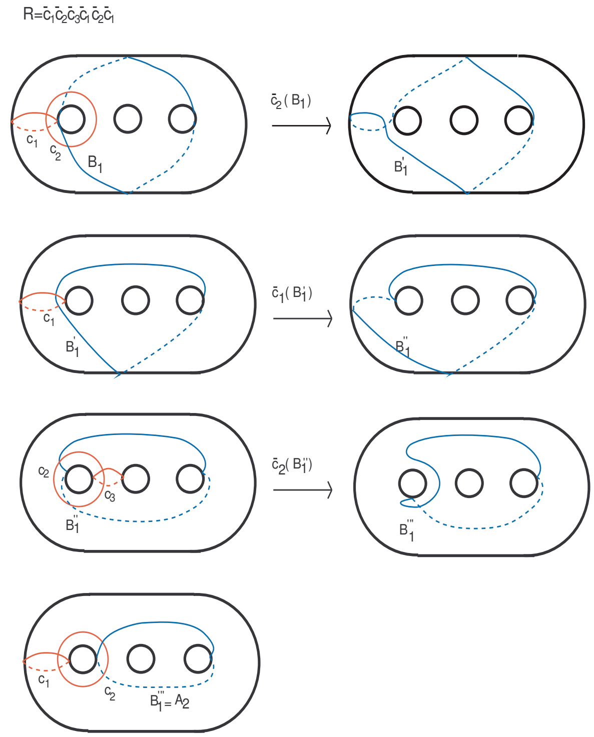

Figure 10. $R(B_1) = A_2$ in genus 3 surface

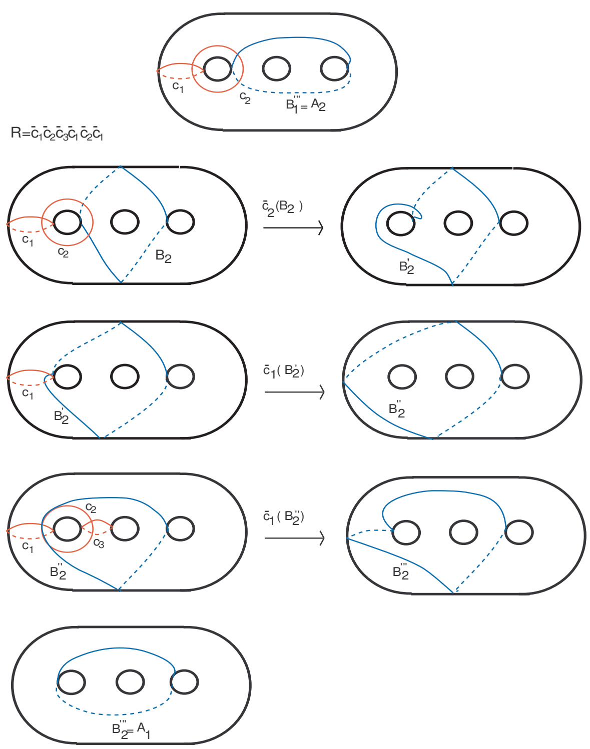

Figure 11. $R(B_2) = A_1$ in genus 3 surface.

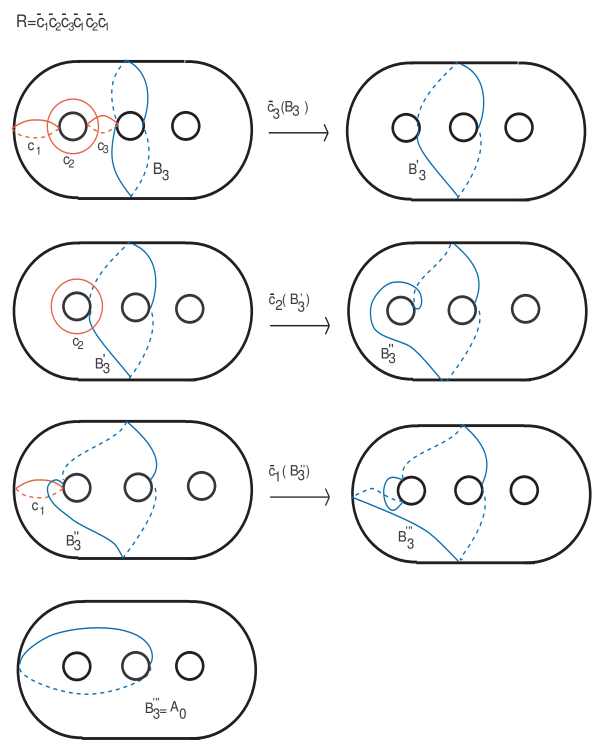

Figure 12. $R(B_3) = A_0$ in genus 3 surface.

Then one can show that $R(B_i) = A_{g-i}$ for all $0 \leq i \leq g$, see Figures 9, 10, 11, and 12 for $R(B_0) = A_3$, $R(B_1) = A_2$, $R(B_2) = A_1$ and $R(B_3) = A_0$ when $g = 3$.

By the above observation we can say that the Dehn twists about the curves A_{g-i} are conjugate with the Dehn twists about the curves B_i, i.e. $A_{g-i} = RB_i\overline{R}$, where A_{g-i} and B_i represent Dehn twists about the corresponding curves. Note also that $R(\sigma) = \sigma$, and $R(a) = a$, $R(b) = b$. Now it is easy to see that the relations of our Main Theorem derives the relations given in Theorem 3.4 of [3].

$$(A_g A_{g-1} \ldots A_0 \sigma)^2 = (RB_0B_1 \ldots B_g \sigma \overline{R})^2 = R(B_0B_1 \ldots B_g\sigma)^2\overline{R} = 1 \quad \text{for } g \text{ is even,}$$

$$(A_g A_{g-1} \ldots A_0 a^2b^2)^2 = (RB_0B_1 \ldots B_g a^2b^2\overline{R})^2 = R(B_0B_1 \ldots B_g a^2b^2)^2\overline{R} = 1 \quad \text{for } g \text{ is odd.}$$

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